A Geometric Criterion for the Convexity of Powell-Sabin Interpolants and its Multivariate Generalization

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Abstract

We derive a geometric criterion for the convexity of Powell-Sabin interpolants and present a multivariate generalization.

Key words: Powell-Sabin interpolant, convex interpolation, multivariate convex C^1-Interpolation, generalized Powell-Sabin elements.

1 Introduction

So far, no construction seems to be known of differentiable, convex, piecewise polynomial functions over $\mathbb{R}^2$ that interpolate any given differentiable convex functions and their gradients over any given set of interpolation abscissae.

In 1992 Carnicer and Dahmen (Carnicer & Dahmen 1992) made an attempt to understand under which additional assumptions one can construct a convex Powell-Sabin interpolant. However, in 1997 Floater (Floater 1997) proved by a counterexample that the additional assumptions of Carnicer and Dahmen do not guarantee convexity or concavity, see also Figure 3.

The purpose of this paper is to give a necessary and sufficient condition for the convexity of the Powell-Sabin interpolant used in (Carnicer, Dahmen 1992). In Section 2 we will show that the convexity depends simply on whether two triangles are disjoint. In the Sections 3 and 4 we present a multivariate generalization of the Powell-Sabin interpolant and of this convexity criterion.

2 Convexity of Powell-Sabin interpolants

Since a Powell-Sabin interpolant is continuously differentiable (Powell & Sabin 1977), it is convex if all its elements are convex. Hence it suffices to discuss the construction by Carnicer and Dahmen for a single Powell-Sabin element. Throughout the paper we will use hollow letters to denote points in $\mathbb{R}^3$ (or later in $\mathbb{R}^n$) and the same letters in bold and normal face to denote their abscissae in $\mathbb{R}^2$ (or $\mathbb{R}^{n-1}$) and (last co)ordinates, respectively. So let

$$\mathbb{p}_i = \begin{bmatrix} \mathbf{p}_i \\ p_i \end{bmatrix} := \begin{bmatrix} \mathbf{p}_i \\ f(\mathbf{p}_i) \end{bmatrix}, \quad i = 1,2,3,$$

be three data points in $\mathbb{R}^3$ sampled from a convex function f and let $\mathcal{P}_i$ be the tangent plane of the graph $[\mathbf{x}^t \ \ f(\mathbf{x})]^t$ at $\mathbb{p}_i$.

As in (Carnicer & Dahmen 1992) we assume that f is **well-associated** with the **data triangle** $\mathbb{p}_1\mathbb{p}_2\mathbb{p}_3$ meaning that $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ intersect in a point $\mathbb{p}_{123}$ whose abscissa $\mathbf{p}_{123}$ lies inside the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$. Since f restricted to any line $\mathbb{p}_i\mathbb{p}_j$ is convex, any two planes $\mathcal{P}_i$ and $\mathcal{P}_j$ have a common point $\mathbb{p}_{ij}$ whose abscissa lies between $\mathbf{p}_i$ and $\mathbf{p}_j$. Obviously, not all convex functions are well-associated with $\mathbb{p}_1\mathbb{p}_2\mathbb{p}_3$.

To state our result we need the three intersections $\mathbb{t}_i$ of the data plane $\mathbb{p}_1\mathbb{p}_2\mathbb{p}_3$ with the lines $\mathcal{P}_j\mathcal{P}_k$, which are the intersections of $\mathcal{P}_j$ and $\mathcal{P}_k$, where (i,j,k) is a cyclic permutation of $(1,2,3)$. These points $\mathbb{t}_i$ form the **tangent triangle**. It is shown in Figures 1, 2, 3 and 5.

Theorem 2.1. *Assume f and the data triangle $\mathbb{p}_1\mathbb{p}_2\mathbb{p}_3$ are well-associated and let $p(\mathbf{x})$ be the Powell-Sabin element that is quadratic over each of the six triangles $\mathbf{p}_i\mathbf{p}_{ij}\mathbf{p}_{123}$, $1 \le i < j \le 3$, interpolating f and its gradient at $\mathbf{p}_1,\mathbf{p}_2$ and $\mathbf{p}_3$. This C^1-Interpolant $p(\mathbf{x})$ is convex if and only if the data triangle $\mathbb{p}_1\mathbb{p}_2\mathbb{p}_3$ lies inside the tangent triangle $\mathbb{t}_1\mathbb{t}_2\mathbb{t}_3$ as illustrated in Figure 3 by two examples.*

Proof:
We need to show that all six quadratic segments of the Powell-Sabin element lie on oval quadrics. Since a quadric is oval if and only if any tangent plane intersects the quadric in two complex conjugate lines (see, e.g., (Boehm & Prautzsch 1994), p. 141), we can use the following property to show convexity.

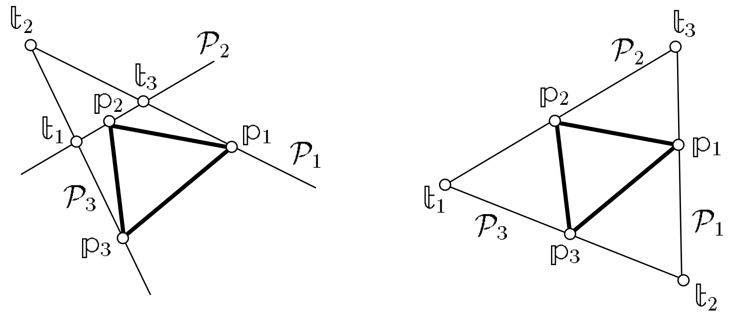

Fig. 1. Data triangle in- and outside the tangent triangle, respectively.

A quadric is oval (not oval) if and only if there is a tangent plane and a conic section on the quadric without (with two) real intersections.

To continue with the proof let $\mathcal{P}_{123}$ be the tangent plane of the Powell-Sabin element $p(\mathbf{x})$ at $\mathbf{x} = \mathbf{p}_{123}$. It passes through the midpoints between $\mathbb{p}_{123}$ and $\mathbb{p}_1$,$\mathbb{p}_2$ and $\mathbb{p}_3$. Thus it is the midplane between $\mathbb{p}_{123}$ and the data plane $\mathbb{p}_1\mathbb{p}_2\mathbb{p}_3$ and thus parallel to the data plane. This is illustrated in Figure 2.

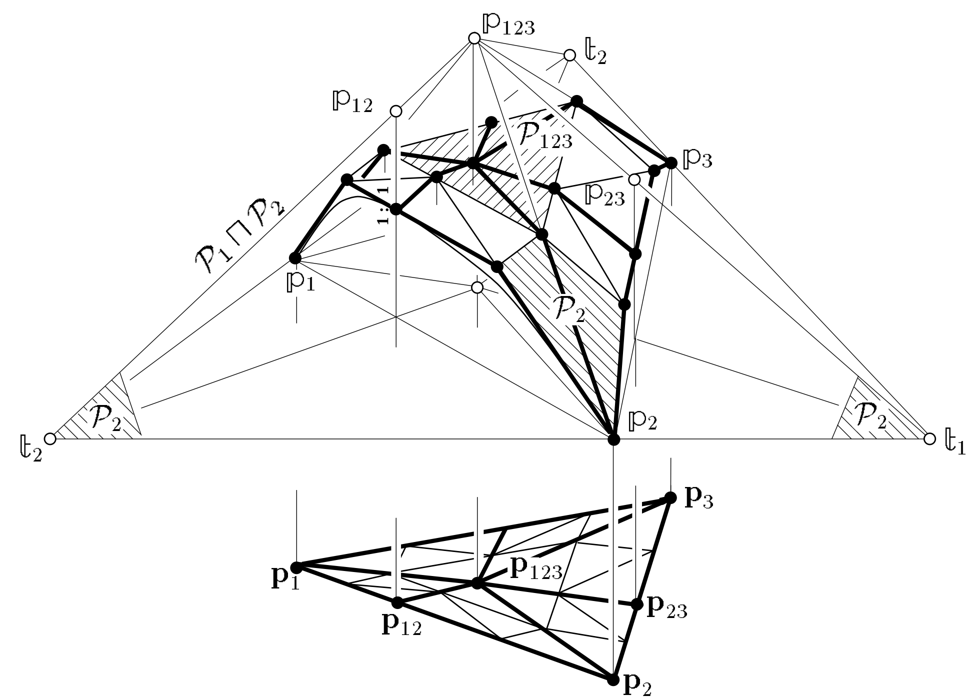

Fig. 2. Construction of a Powell-Sabin element for a concave function f.

Further the Powell-Sabin element restricted to any line $\mathbf{p}_i\mathbf{p}_j$ lies on two convex (i.e., not concave) parabolas which are tangent by construction to the midline $\mathcal{T}_{ij}$ between $\mathbb{p}_{ij}$ and the line $\mathbb{p}_i\mathbb{p}_j$.

This tangent $\mathcal{T}_{ij}$ is parallel to $\mathcal{P}_{123}$. Therefore both parabolas lie above $\mathcal{P}_{123}$ if and only if the tangent $\mathcal{T}_{ij}$ does so too. If $\mathcal{P}_i \sqcap \mathcal{P}_j$ is parallel to $\mathcal{P}_{123}$, then $\mathbb{t}_k$ is at infinity and the tangent $\mathcal{T}_{ij}$ lies on $\mathcal{P}_{123}$. If $\mathbf{t}_k$ and $\mathbf{p}_{123}$ lie on opposite sides of the edge $\mathbf{p}_i\mathbf{p}_j$ then $\mathcal{T}_{ij}$ lies below. So, both parabolas do not intersect $\mathcal{P}_{123}$ in a real point if and only if the edge $\mathbf{p}_i\mathbf{p}_j$ separates $\mathbf{p}_{123}$ and $\mathbf{t}_k$. Consequently all six quadratic segments of the Powell-Sabin element are oval if and only if the triangle $\mathbb{p}_1\mathbb{p}_2\mathbb{p}_3$ lies inside $\mathbb{t}_1\mathbb{t}_2\mathbb{t}_3$ which concludes the proof. $\square$

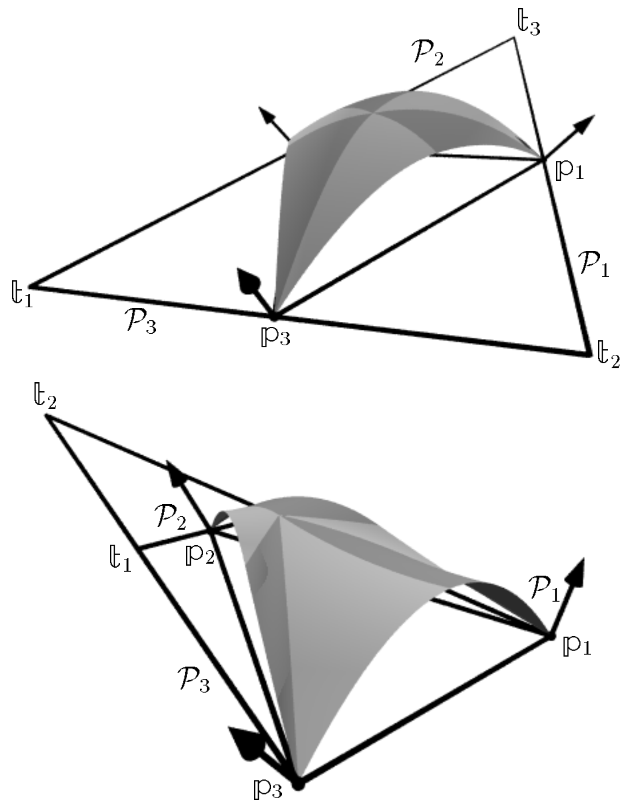

Fig. 3. Powell-Sabin-Interpolants to strictly concave data with "well associated" tangent planes.

Willemans and Dierckx (Willemans & Dierckx 1994) expressed the convexity of a general Powell-Sabin element by twelve inequalities to build convex approximants. Since the convexity of each quadratic segment is characterized by one inequality, six inequalities would suffice. Moreover for the special situation of the theorem above we can reduce the number further to three inequalities.

Corollary 2.2. *Let $d(\mathcal{P}_i, \mathbb{p}_j)$ denote the Euclidean distance between the plane $\mathcal{P}_i$ and the point $\mathbb{p}_j$ and let*

$$g_{ijk} \ := \ d(\mathcal{P}_i, \mathbb{p}_j)\ d(\mathcal{P}_j, \mathbb{p}_k) + d(\mathcal{P}_i, \mathbb{p}_k)\ d(\mathcal{P}_j, \mathbb{p}_i) - d(\mathcal{P}_i, \mathbb{p}_j)\ d(\mathcal{P}_j, \mathbb{p}_i),$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$. Then, the tangent triangle $\mathbb{t}_1\mathbb{t}_2\mathbb{t}_3$ contains the data triangle $\mathbb{p}_1\mathbb{p}_2\mathbb{p}_3$ if and only if g_{123}, g_{231} and g_{312} are all positive.

Proof:
Observe that g_{ijk} depends linearly on $\mathbb{p}_k$ since the Euclidean distance between a point and a plane can be expressed by a scalar product. Further, g_{ijk} is zero for $\mathbb{p}_k$ on the edge $\mathbb{p}_i\mathbb{p}_j$ and $g(\mathbb{p}_k) < 0$ for $\mathbb{p}_k = \mathbb{t}_k$. Hence g_{ijk} is positive if and only if the edge $\mathbb{p}_i\mathbb{p}_j$ separates $\mathbb{p}_k$ and $\mathbb{t}_k$. The other two inequalities follow analogously. $\square$

Given a convex triangular net with vertices $\mathbb{p}_i$, $i = 1,\ldots,n$, there is no guarantee that well-associated tangent planes $\mathcal{P}_i$ exist. Moreover, it is not clear how to find well-associated tangent planes if they exist.

Anyhow, there are simple inequalities which imply or rule out well-associatedness if satisfied or not satisfied, see (Carnicer & Dahmen '92). These inequalities are as follows. Let $T := \{(i,j,k) \mid \mathbb{p}_i\mathbb{p}_j\mathbb{p}_k$ is a triangle of the net$\}$ and let

$$\mathcal{P}_i(\mathbb{x}) \;=\; \begin{bmatrix} -\nabla f(\mathbf{p}_i) \\ 1 \end{bmatrix} \left(\mathbb{x} \;-\; \begin{bmatrix} \mathbf{p}_i \\ f(\mathbf{p}_i) \end{bmatrix} \right) \;=\; 0$$

denote the equation of the tangent plane at $\mathbb{p}_i$, where $\nabla f(\mathbf{p}_i)$ is the gradient of f. Then the points $\mathbb{p}_i$ and tangent planes $\mathcal{P}_i$ belong to a convex function if and only if for all edges $\mathbb{p}_i\mathbb{p}_j$ of the triangular net

$$a_{ij} \;:=\; \mathcal{P}_i(\mathbb{p}_j) \;>\; 0.$$

Further, they are well associated if and only if for all $(i,j,k) \in T$

$$d_{ijk} \;:=\; a_{ij}a_{jk} + a_{ik}a_{kj} - a_{kj}a_{jk} \;>\; 0.$$

Similarly we can express whether the tangent triangle contains the data triangle. According to Corollary 2.2, the tangent triangle contains the data triangle if and only if for all $(i,j,k) \in T$

$$b_{ijk} \;:=\; a_{ij}a_{jk} + a_{ik}a_{ji} - a_{ij}a_{ji} > 0,$$

namely b_{ijk} is a positive multiple of g_{ijk}. Note that a_{ij} depends linearly on $\mathcal{P}_i$. Therefore d_{ijk} and b_{ijk} depend quadratically on the $\mathcal{P}$'s.

Carnicer and Dahmen propose to compute tangent planes $\mathcal{P}_i$ that minimize the sum

$$\sum_{(i,j,k)\in T} \left(\alpha_{ij}(a-a_{ij})^2 + \alpha_{jk}(a-a_{jk})^2 + \alpha_{ki}(a-a_{ki})^2 \;+\; \delta_{ijk}(d-d_{ijk})\right),$$

where the α's and δ's are certain weights and a and d are some constants. This sum could be extended by the terms $\beta_{ijk}(b-b_{ijk})$.

3 A multivariate generalization of the Powell-Sabin element

In this section we present a multivariate generalization of the (bivariate) Powell-Sabin interpolant. And in the next section we derive for this interpolant a convexity criterion which generalizes Theorem 2.1.

Given a simplex σ in $\mathbb{R}^{n-1}$ with vertices $\mathbf{p}_1,\ldots,\mathbf{p}_n$ we choose on each face $\mathbf{p}_{i_1}\ldots\mathbf{p}_{i_m}$ of σ, where $I := \{i_1,\ldots,i_m\} \subset \{1,\ldots,n\}$, some (splitting) point $\mathbf{p}_{i_1\ldots i_m}$. Note that $\mathbf{p}_{i_1\ldots i_m}$ has different notations. Namely for all permutations $(j_1,\ldots,j_m)$ of $(i_1,\ldots,i_m)$ we set $\mathbf{p}_{j_1\ldots j_m} = \mathbf{p}_{i_1\ldots i_m}$, and also use the notation $\mathbf{p}_I$.

Lemma 3.1. *Let S_n be the symmetric group of all permutations of $(1,\ldots,n)$ and let $\pi = (\pi_1,\ldots,\pi_n) \in S_n$. Further we denote the tails of π by*

$$I_k := I_k(\pi) := \{\pi_k,\ldots,\pi_n\}$$

and the simplex $\mathbf{p}_{I_1}\ldots\mathbf{p}_{I_n}$ by σ_π. Then the simplices σ_π, $\pi \in S_n$ form a partition of σ.

Remark 3.2. *For $n=3$ the simplices σ_π form a Powell-Sabin split of σ, see Figure 4.*

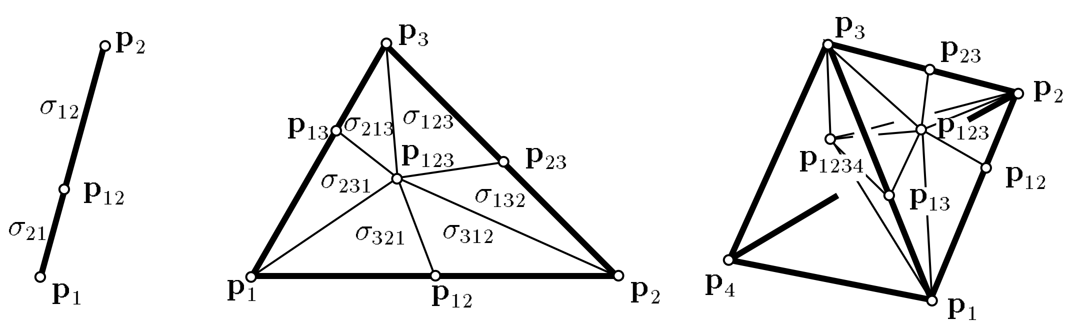

Fig. 4. A partition of the simplex σ for $n = 2, 3, 4$.

Proof:
We prove the lemma by induction over n. For $n=2$ and $n=3$ the lemma is obviously true, see also Figure 4. Next we assume that the lemma holds if σ had only $n-1$ vertices, where $n \geq 2$. Consequently, all simplices $\mathbf{p}_{I_2} \dots \mathbf{p}_{I_n}$, $\pi \in S_n$ where π_1 is fixed, form a partition of the face $\mathbf{p}_{\pi_2} \dots \mathbf{p}_{\pi_n}$ of σ and the simplices σ_π form a partition of the simplex σ_{π_1} obtained from σ by replacing the vertex $\mathbf{p}_{\pi_1}$ by $\mathbf{p}_{1\dots n}$. Figure 4 shows for $n = 4$ the simplices σ, σ_4 and σ_π, where $\pi_1 = 4$. Since the n simplices $\sigma_1, \dots, \sigma_n$ form a partition of σ, the lemma also holds for a simplex with n vertices. This concludes the proof.$\square$

Next we show that there is a unique C^1-function $p(\mathbf{x})$ over σ that is quadratic over each simplex σ_π and interpolates any function f and its gradients at the n vertices $\mathbf{p}_1, \dots, \mathbf{p}_n$ of σ.

We will use the Bézier representation of $p(\mathbf{x})$. So let Δ be the set of all multiindices $\mathbb{i} = (i_1, \dots, i_n) \in \{0, 1, 2\}^n$, where $|\mathbb{i}| = i_1 + \dots + i_n = 2$ and let $b_{\pi\mathbb{i}}$ be the Bézier ordinates of $p(\mathbf{x})$ over σ_π. This means that

$$p(\mathbf{x}) = \sum_{\mathbb{i} \in \Delta} b_{\pi\mathbb{i}} \, B^2_{\mathbb{i}}(\mathbb{u}),$$

where

$$B^2_{\mathbb{i}}(\mathbb{u}) = \binom{n}{\mathbb{i}} \mathbb{u}^{\mathbb{i}} = \frac{n!}{i_1! \dots i_n!} \, u_1^{i_1} \dots u_n^{i_n}$$

are the quadratic Bernstein polynomials and $u_1, \dots, u_n$ are the barycentric coordinates of $\mathbf{x}$ with respect to $\mathbf{p}_{I_1} \dots \mathbf{p}_{I_n}$.

Further let $\mathbb{b}_{\pi,\mathbb{i}} = [\mathbf{b}^t_{\pi,\mathbb{i}} \quad b_{\pi,\mathbb{i}}]^t$ be the Bézier points of the graph $[\mathbf{x}^t \quad p(\mathbf{x})]^t$. Their abscissae are given by, see e.g. (Prautzsch, Boehm '99, 10.3 and 19.3),

$$\mathbf{b}_{\pi,\mathbb{i}} = \frac{1}{2} \sum_{k=1}^{n} i_k \mathbf{p}_{I_k}.$$

Note that either two coordinates i_k are one or one i_k equals two while all other i_k are zero.

Lemma 3.3. *Given a differentiable function f over σ there are unique ordinates $b_{\pi\mathbb{i}}$ such that the piecewise quadratic $p(\mathbf{x})$ is differentiable and interpolates f and its derivatives at the vertices of σ.*

Proof:
We prove the lemma by induction over n. For $n=2,3$ the lemma is well-known. So we assume that the lemma holds for any $n-1$ dimensional simplex, where $n\geq 3$.

Hence we can determine all ordinates $q_{\pi,\mathring{\mathbb{i}}}$, where $i_1 = 0$, over the faces of σ such that $p(\mathbf{x})$ restricted to any face of σ satisfies the lemma.

Then we set all ordinates $b_{\pi,10\ldots01}$ so that $p(\mathbf{x})$ has the same derivatives as $f(\mathbf{x})$ at $\mathbf{p}_{\pi_n} = \mathbf{b}_{\pi,0\ldots02}$. Finally we choose all remaining ordinates $b_{\pi,\mathring{\mathbb{i}}}, \pi \in S_n, \mathring{\mathbb{i}} \in \Delta$, where $i_1 \geq 1$, such that all inner points $\mathbb{b}_{\pi,\mathring{\mathbb{i}}}$ lie on the hyperplane spanned by the n points $\mathbb{b}_{\pi,10\ldots01}, \pi_1 = 1,\ldots,n$. Hence all quadratic segments of $p(\mathbf{x})$ have C^1-contact at all points $\mathbf{p}_I, I \subset \{1,\ldots,n\}$.

Therefore, all derivatives of $p(\mathbf{x})$, which are linear over each subsimplex σ_π, are continuous. This concludes the proof. $\square$

Consider the piecewise quadratic interpolant $p(\mathbf{x})$ over the simplices σ_π, where $\pi_1 = 1$. The derivatives of $p(\mathbf{x})$ are continuous. In particular, this is true over the edge $\mathbf{p}_{1\ldots n}\mathbf{p}_{2\ldots n}$. Hence any second derivative of $p(\mathbf{x})$ with respect to the direction $\mathbf{p}_{1\ldots n}-\mathbf{p}_{2\ldots n}$ and an arbitrary second direction is also continuous over this edge and thus constant rather than piecewise constant. Consequently the directional derivative of $p(\mathbf{x})$ with respect to the difference $\mathbf{p}_{1\ldots n} - \mathbf{p}_{2\ldots n}$ is linear over the entire face $\mathbf{p}_2 \ldots \mathbf{p}_n$ rather than piecewise linear.

Thus we obtain the following result:

Corollary 3.4. *Let $\bar{p}(\mathbf{x})$ denote the piecewise quadratic C^1-interpolant constructed as in Lemma 3.3 for the same function f over an adjacent simplex $\bar{\sigma}$ with vertices $\bar{\mathbf{p}}_1,\ldots,\bar{\mathbf{p}}_n$, where $\mathbf{p}_k = \bar{\mathbf{p}}_k$ for $k \geq 2$, $\mathbf{p}_{2\ldots n} = \bar{\mathbf{p}}_{2\ldots n}$ and $\mathbf{p}_{1\ldots n}$, $\mathbf{p}_{2\ldots n}$, $\bar{\mathbf{p}}_{1\ldots n}$ are collinear. Then $p(\mathbf{x})$ and $\bar{p}(\mathbf{x})$ have C^1-contact over the common face $\mathbf{p}_2 \ldots \mathbf{p}_n$.*

4 The multivariate convexity criterion

Let σ be a simplex in $\mathbb{R}^{n-1}$ with vertices $\mathbf{p}_1,\ldots,\mathbf{p}_n$ and let $\sigma_I, I \subset \{1,\ldots,n\}$, be the face of σ with vertices $\mathbf{p}_i, i \in I$. Further let f be a differentiable convex function defined on σ and let $\mathcal{P}_i$ be the tangent hyperplane in $\mathbb{R}^n$ of the graph $[\mathbf{x}^t\ f(\mathbf{x})]^t$ at $\mathbf{x} = \mathbf{p}_i$. In the sequel we drop the suffix “hyper”.

We assume that the tangent planes $\mathcal{P}_i$ are in general positon meaning that for all subsets $I \subset \{1, \dots, n\}$ there is exactly one point $\mathbb{p}_I$ in all planes $\mathcal{P}_i$, $i \in I$, whose abscissa, $\mathbf{p}_I$, lies in the affine hull of σ_I.

Hence the planes $\mathcal{P}_i$ partition $I\!R^n$ into 2^n regions in the same way as the n coordinate planes $x_i = 0$ do. One of these regions, we call it $\mathcal{S}$, contains all the **data points** $\mathbb{p}_i = [\mathbf{p}_i{}^t\ f(\mathbf{p}_i)]^t$. It is the intersection of all half spaces $\mathcal{H}_i$ containing the points on or above $\mathcal{P}_i$. The infinite pyramid $\mathcal{S}$ has the apex $\mathbb{p}_{1\dots n}$ and n edges, where each edge lies in all but one of the half spaces $\mathcal{H}_i$. To prove the convexity criterion below in Theorem 4.2 we need the following fact:

Lemma 4.1. *If $\mathbf{p}_{2\dots n}$ lies in the face $\sigma_{2\dots n}$, then $\mathbb{p}_{2\dots n}$ lies on the edge $\mathcal{H}_1 \cap \mathcal{P}_2 \cap \dots \cap \mathcal{P}_n$ of $\mathcal{S}$ as illustrated in Figure 5.*

Proof:
Let $\mathbb{e}_1, \dots, \mathbb{e}_n$ be positive or negative multiples of the n directions $\mathbb{v}_i := \mathbb{p}_{\{1\dots n\}\setminus\{i\}} - \mathbb{p}_{1\dots n}$, such that

$$\mathcal{S} \; = \; \{\mathbb{p}_{1\dots n} \; + \; \sum_{i=1}^{n} \epsilon_i \mathbb{e}_i | \; \epsilon_1, \dots, \epsilon_n \geq 0\}.$$

We need to show that $\mathbb{e}_1$ is a positive multiple of $\mathbb{v}_1$.

Let $\mathbb{u}_i := \mathbb{p}_i - \mathbb{p}_{1\dots n}$. Since $\mathbb{p}_i \in \mathcal{S}$, there is a non-negative matrix U such that

$$[\mathbb{u}_1 \dots \mathbb{u}_n] \; = \; [\mathbb{e}_1 \dots \mathbb{e}_n]U.$$

Further, since $\mathbf{p}_{2\dots n}$ is a convex combination of $\mathbf{p}_2 \dots \mathbf{p}_n$, there is a non-negative column $\mathbb{c}$ such that

$$\begin{aligned} \mathbf{v}_1 &= [\mathbf{u}_1 \dots \mathbf{u}_n]\mathbb{c} \\ &= [\mathbf{e}_1 \dots \mathbf{e}_n]U\mathbb{c}. \end{aligned}$$

Hence $\mathbf{v}_1$ and $\mathbb{v}_1$ are positive multiples of $\mathbf{e}_1$ and $\mathbb{e}_1$, respectively, which concludes the proof. $\square$

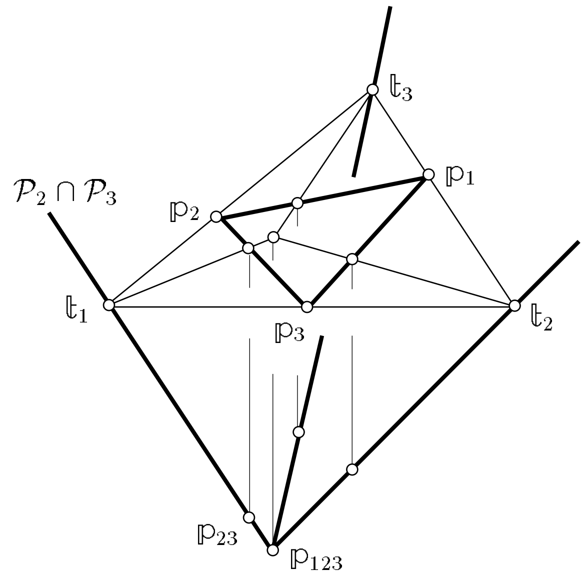

Fig. 5. The Pyramid $\mathcal{S}$.

If all points $\mathbf{p}_I, I \subset \{1, \ldots, n\}$, lie in the interior of the corresponding simplex face σ_I, then we call f and σ **well-associated**. Note that not all convex functions are well-associated with σ.

Finally let $\mathbb{t} = [\mathbf{t}_i^t \; t_i]^t$ be the intersection of the $n-1$ hyperplanes $\mathcal{P}_j$, $j \neq i$, with the **data plane** $\mathbb{p}_1 \ldots \mathbb{p}_n$. We call the simplices $\mathbb{t}_1 \ldots \mathbb{t}_n$ and $\mathbb{p}_1 \ldots \mathbb{p}_n$ the **tangent** and **data simplex** respectively.

With these definitions the generalization of Theorem 2.1 is straightforward:

Theorem 4.2. *Assume f and σ are well-associated and let $p(\mathbf{x})$ be the piecewise quadratic interpolant to f constructed in Lemma 3.3 with respect to the above splitting points $\mathbf{p}_I$. Then $p(\mathbf{x})$ is convex if and only if $p(\mathbf{x})$ is convex over all faces $\sigma_I, |I| = n-1$, and the tangent simplex contains the data simplex.*

Remark 4.3. *If $n = 2$, then σ is well-associated with every convex function f. Further, the tangent simplex contains the data simplex and the univariate interpolant $p(x)$ is convex over the end points of σ. Hence $p(x)$ is always convex for all convex functions f.*

Consequently, if $n = 3$, then $p(\mathbf{x})$ is convex over all edges σ_I, $|I| = n-1$ for all convex functions f. Therefore Theorem 2.1 needs one assumption less than Theorem 4.2.

Proof of Theorem 4.2:
A function is convex if and only if its restriction to any line is convex. Since a quadratic polynomial has constant second derivatives, it is convex if its restrictions to all lines through a fixed point are convex.

Hence a quadratic polynomial $q(\mathbf{x})$ is convex if and only if its restriction to any hyperplane $\mathcal{H}$ is convex and lies above the tangent plane of $q(\mathbf{x})$ at some point not in $\mathcal{H}$.

Now consider, for example, the subsimplex $\sigma_{1\ldots n}$ and let $q(\mathbf{x})$ be the quadratic polynomial that equals the interpolant $p(\mathbf{x})$ over $\sigma_{1\ldots n}$. Due to the construction of $p(\mathbf{x})$ the tangent plane $\mathcal{P}_{1\ldots n}$ of $p(\mathbf{x})$ at $\mathbf{x} = \mathbf{p}_{1\ldots n}$ is the midplane between the point $\mathbb{p}_{1\ldots n}$ and the data plane $\mathbb{p}_1 \ldots \mathbb{p}_n$.

Similarly the tangent plane at $\mathbf{p}_{2\ldots n}$ of $q(\mathbf{x})$ and $p(\mathbf{x})$ restricted to $\sigma_{2\ldots n}$ is the midplane between $\mathbb{p}_{2\ldots n}$ and the plane $\mathbb{p}_2 \ldots \mathbb{p}_n$. It is parallel to the tangent plane $\mathcal{P}_{1\ldots n}$ and lies above $\mathcal{P}_{1\ldots n}$ if and only if $\mathbb{p}_{2\ldots n}$ separates $\mathbb{p}_{1\ldots n}$ and $\mathbb{t}_1$, the intersection of the line $\mathcal{P}_2 \cap \ldots \cap \mathcal{P}_n$ with the data plane. Note that both $\mathbb{p}_{1\ldots n}$ and $\mathbb{p}_{2\ldots n}$ lie below the data plane. Now, as a consequence of Lemma 4.1, $q(\mathbf{x})$ is convex if and only if $\mathbb{t}_1$ lies on an edge of the pyramid $\mathcal{S}$.

Analogous properties hold for the other quadratic segments of the interpolant $p(\mathbf{x})$. Thus $p(\mathbf{x})$ is convex if and only if the intersection of $\mathcal{S}$ with the data plane is the tangent simplex $\mathbb{t}_1 \ldots \mathbb{t}_n$.

This finishes the proof since $\mathcal{S}$ and the data plane contain σ and since the tangent simplex always lies in one of the 2^n disjoint pyramids formed by the planes $\mathcal{P}_i$. $\square$

The convexity criterion can also be expressed by n inequalities which we have seen already in the bivariate case.

Corollary 4.4. *Let d_{ij} denote the Euclidean distance between the plane $\mathcal{P}_i$ and the point $\mathbb{p}_j$ and let D be the $n \times n$-matrix $[d_{ij}]_{i,j=1}^n$. Further let A_i and B_i denote the matrices obtained from the $n \times n$-matrix $[d_{ij}]_{i,j=1}^n$ by replacing the i-th row by the row $[1 \ \ldots \ 1]$ or by deleting the i-th row and i-th column, respectively.*

If f and σ are well-associated, then the tangent simplex contains the data *simplex if and only if for all* $i = 1, \ldots, n$

$$g_i \ := \ \det A_i \ \cdot \ \det B_i \ < \ 0.$$

Proof:

The functions g_i depend linearly on $\mathbb{p}_i$ since the Euclidean distance between a point and a plane can be expressed by a scalar product, and it holds

$$g_i = \begin{cases} 0 & \text{for } \mathbb{p}_i = \mathbb{p}_j,\ j \neq i \\ (\det B_i)^2 & \text{for } \mathbb{p}_i = \mathbb{t}_i. \end{cases}$$

Since D is the product of two matrices with full rank n,

$$D \ = \ \begin{bmatrix} \mathbb{w}_1 & \ldots & \mathbb{w}_n \\ w_{10} & \ldots & w_{n0} \end{bmatrix}^t \begin{bmatrix} \mathbb{p}_1 & \cdots & \mathbb{p}_n \\ 1 & \ldots & 1 \end{bmatrix},$$

the submatrices B_i are regular. So g_i is negative if and only if $\mathbb{p}_i$ and $\mathbb{t}_i$ lie on opposite sides of the face $\sigma_{\{1\ldots n\}\setminus\{i\}}$. As we have seen in the proof of Theorem 4.2 this is all we need to show. □

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