Geometric Fundamentals

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Abstract

The following recalls the facts and terminology mostly used in Geometry. It may serve also as a first introduction to geometric tools, for more in depth coverage see the list of references, in particular [7].

1 Affine Fundamentals

Many properties of computational geometry and its applications do not need the distance of points but only the concepts of parallelism and ratio. Without additional effort most of the following topics can be studied in spaces of any dimension.

1.1 Points and Vectors

In general a point in n-space is fixed by its coordinates with respect to some Cartesian system. Nevertheless, we start our observations with affine aspects[1].

[1] Matrix notation is preferred. To simplify the notation, a point as well as its coordinate column will be denoted by the same bold letter. Note that this notation depends on the coordinate system.

Let $\mathbf{a} = [\alpha_1 \dots \alpha_n]^t$ and $\mathbf{b} = [\beta_1 \dots \beta_n]^t$ denote two points. Their difference $\mathbf{v} = \mathbf{b} - \mathbf{a}$ is called a vector, and one has $\mathbf{b} = \mathbf{a} + \mathbf{v}$. In particular the column $\mathbf{o} = [\,0 \dots 0\,]^t$ denotes the null-vector.

Let $\mathbf{a}_0, \dots, \mathbf{a}_d$ denote $d+1$ points in n-space, $d \le n$. The d vectors $\mathbf{v}_i = \mathbf{a}_i - \mathbf{a}_0$, $i = 1, \dots, d$, are called linearly dependent if $\alpha_1 \mathbf{v}_1 + \dots + \alpha_d \mathbf{v}_d = \mathbf{o}$, with at least one non-zero α_i, otherwise these vectors are called linearly independent. If $\mathbf{v}_1, \dots, \mathbf{v}_d$ are linearly independent, then they span a linear space $\mathbf{V}^d$, and the points $\mathbf{a}_0, \dots, \mathbf{a}_d$ are called affinely independent and span an affine space $\mathbf{A}^d$ of dimension d.

1.2 Affine Systems

A point $\mathbf{a}_0$ and n linearly independent vectors $\mathbf{v}_i$ define an affi= ne system $[\mathbf{a}_0, \mathbf{v}_1 \dots \mathbf{v}_n]$ of $\mathbf{A}^n$. In this system every point $\mathbf{p} = [\eta_1 \dots \eta_n]^t$ may be written uniquely as

$$\mathbf{p} = \mathbf{a}_0 + \xi_1 \mathbf{v}_1 + \dots + \xi_n \mathbf{v}_n = \mathbf{a}_0 + A\mathbf{x}, \tag{1}$$

where $\mathbf{A} = [\mathbf{v}_1 \dots \mathbf{v}_n]$ and $\mathbf{x} = [\xi_1 \dots \xi_n]^t$. The ξ_i are called the affine coordinates of $\mathbf{p}$ with respect to $[\mathbf{a}_0, \mathbf{v}_1 \dots \mathbf{v}_n]$, where $\mathbf{a}_0$ is called the origin of the affine system.

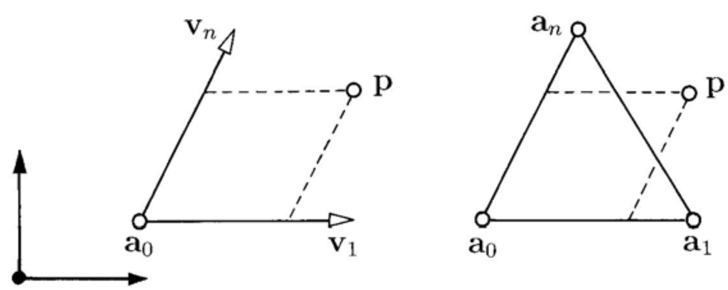

Figure 1: Affine and barycentric coordinates.

To distinguish between points and vectors described by elements $\mathbf{a}$ of $\mathbb{R}^n$ one may add a further coodinate, ϵ, where

$$\epsilon = \begin{cases} 0 \\ 1 \end{cases} \text{if } \mathbf{a} \text{ represents a } \begin{cases} \text{vector,} \\ \text{point.} \end{cases}$$

This convention can help to avoid errors in handling points and vectors, see also Subsection **4.1** on homogeneous coordinates.

1.3 Barycentric Coordinates

Let $\mathbf{a}_0, \ldots, \mathbf{a}_n$ denote $n+1$ affinely independent points of $\mathbf{A}^n$ and let $\mathbf{v}_i = \mathbf{a}_i - \mathbf{a}_0$. One may rewrite equation (1) as

$$\mathbf{p} = \xi_0 \mathbf{a}_0 + \xi_1 \mathbf{a}_1 + \cdots + \xi_n \mathbf{a}_n, \quad \xi_0 = 1 - (\xi_1 + \cdots + \xi_n).$$

The $\xi_0, \ldots, \xi_n$ are called barycentric coordinates of $\mathbf{p}$ with respect to the frame $[\mathbf{a}_0 \ldots \mathbf{a}_n]$. Note that $\xi_0 + \cdots + \xi_n = 1$. Note also that any n of the ξ_i, $i \neq j$, represent affine coordinates of $\mathbf{p}$ with respect to an affine system with origin $\mathbf{a}_j$.

Immediately follows that a vector $\mathbf{v} = \mathbf{b} - \mathbf{a}$ has barycentric coordinates $\nu_0, \ldots, \nu_n$ that sum to zero, $\nu_0 + \cdots + \nu_n = 0$. Note that the sum of the coordinates ξ_i or ν_i corresponds to ϵ above.

1.4 Affine Subspaces and Parallelism

Let points $\mathbf{a}_0, \ldots, \mathbf{a}_d \in \mathbf{A}^n$ be given. The point

$$\mathbf{p} = \xi_0 \mathbf{a}_0 + \xi_1 \mathbf{a}_1 + \cdots + \xi_d \mathbf{a}_d, \quad 1 = \xi_0 + \xi_1 + \cdots + \xi_d \,,$$

is called an affine combination of the points $\mathbf{a}_i$. Let $d \leq n$ and let the points $\mathbf{a}_i$ be affinely independent. Then they span a d-dimensional subspace $\subset \mathbf{A}^n$. Using barycentric coordinates, its points are written as affine combination=s (in terms of affine coordinates) as

$$\mathbf{p} = \mathbf{a}_0 + \xi_1 \mathbf{v}_1 + \cdots + \xi_d \mathbf{v}_d \,, \tag{2}$$

where $\mathbf{v}_i = \mathbf{a}_i - \mathbf{a}_0$. This subspace is called a line, plane or hyperplane if $d = 1, 2$ or $n-1$, respectively.

For any given $\mathbf{p}$ and given $\mathbf{a}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d$ as elements of $\mathbb{R}^n$, the linear combination (2) represents a system of n linear equations for the ξ_i. It is solvable only if $\mathbf{p}$ lies in the subspace.

Conversely, for varying ξ_i the presentation (2) can be viewed as the solution of some linear system of $n-d$ equations for an unknown $\mathbf{p} = [\eta_1 \ldots \eta_n]^t$. In particular, if $d = n-1$, this system consists of one linear equation, say

$$u_0 + \mathbf{u}^t \mathbf{p} = u_0 + u_1 \eta_1 + \cdots + u_n \eta_n = 0,$$

or, using barycentric coordinates of $\mathbf{p}$,

$$u_0 \eta_0 + (u_1 + u_0)\eta_1 + \cdots + (u_n + u_0)\eta_n = 0,$$

where additionally $\eta_0 + \cdots + \eta_n = 1$.

Hyperplanes are called linearly independent if the rows $[u_0\, u_1 \ldots u_n]$ of their coefficients are linearly independent. Consequently, a subspace of dimension d of $\mathbf{A}^n$ can be obtained as the intersection of $n-d$ linearly independent hyperplanes. In particular, a point can be obtained as the intersection of n hyperplan= es.

Note that the points of an affine subspace are the solutions of an inhomogeneous linear system, while their differences, the vectors, solve the correspondin= g homogeneous system.

A line $\mathbf{p} = \mathbf{a} + \lambda \mathbf{v}$ is called parallel to a subspace $\mathbf{B} \subset \mathbf{A}^n$ if the coordinates of its points solve th= e homogeneous system corresponding to $\mathbf{B}$. Moreover, two affine subspaces $\mathbf{A}$ and $\mathbf{B}$ are parallel if all lines of $\mathbf{A}$ are parallel to $\mathbf{B}$, or vice versa.

1.5 Affine Maps and Axonometric Images

Equation (1) allows two interpretations. First, it expresses $\mathbf{p}$ with respect to a new affine system $[\mathbf{a}_0, \mathbf{v}_1 \ldots \mathbf{v}_n]$, where $\mathbf{x} = [\xi_1 \ldots \xi_n]^t$ are the new coordinates of $\mathbf{p}$. For example, the equation $u_0 + \mathbf{u}^t\mathbf{p} = 0$ of a hyperplane becomes $q_0 + \mathbf{q}^t\mathbf{x} = 0$ in terms of the new coordinates , where $q_0 = u_0 + \mathbf{u}^t\mathbf{a}_0$ and $\mathbf{q}^t = \mathbf{u}^t A$.

Second, it represents an affine map $\phi : \mathbf{x} \to \mathbf{p}$, where $\mathbf{x}$ and $\mathbf{p}$ represent affine coordinates of two points. In particular, ϕ maps the origin $\mathbf{o} = [0 \ldots 0]$ and the unit vectors[2] $\mathbf{e}_i = [\delta_{i,1} \ldots \delta_{i,n}]^t$ into $\mathbf{a}_0$ and $\mathbf{v}_i$, $i = 1, \ldots, n$, respectively. This important property defines the map uniquely and allows a simple construction and analysis of an affine map. Note that ϕ maps the points $\mathbf{x}$ of the hyperplane $q_0 + \mathbf{q}^t\mathbf{x} = 0$ above into the points $\mathbf{p}$ of the hyperplane $u_0 + \mathbf{u}^t\mathbf{p} = 0$.

If the barycentric coordinate columns of points are denoted by the corresponding hollow letters, ϕ is written as

$$\mathbb{p} = \mathbb{A}\mathbb{x}\,, \quad \text{where } \mathbb{A} = [\mathbb{a}_0\, \mathbb{a}_1 == \ldots \mathbb{a}_n].$$

Note that affine maps preserve affine combinations, i.e. one has

$$\phi[\xi_0\mathbf{a}_0 + \cdots + \xi_d\mathbf{a}_d] = \xi_0[\phi\mathbf{a}_0] + \cdots + \xi_d[\phi{=}a_d].$$

[2] $\delta_{i,j}$ is the so-called Kronecker delta, $\delta_{i,j} = 1$ if $i = j$ and $= 0$ else

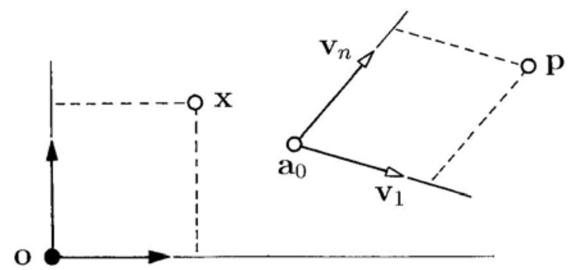

Figure 2: Affine map and new affine system.

They also preserve parallelism and the ratio of parallel distances, i.e. $\mathbf{w} = \lambda \mathbf{v}$ is mapped into $[\phi \mathbf{w}] = \lambda[\phi \mathbf{v}]$.

If the $\mathbf{v}_i$ are linearly dependent, the map is degenerate. In particular, if $\eta_{d+1} = \ldots = \eta_n = 0$ for all $\mathbf{x}$, the map creates an axonometric image as used in descriptive geometry. Simple examples are the so-called cavalier and military projections, see [7].

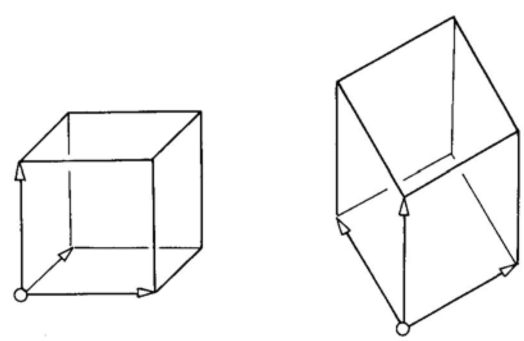

Figure 3: Cavalier and military projection.

1.6 Affine Combinations and A-Frame

Many algorithms in CAGD are based on repeated affine combinations. Consider two points, $\mathbf{a}_0$ and $\mathbf{a}_1$. The affine combination

$$\mathbf{p} = (1 - \alpha)\mathbf{a}_0 + \alpha\, \mathbf{a}_1$$

represents a point on the line spanned by $\mathbf{a}_0$ and $\mathbf{a}_1$, and α represents an affine scale with $\alpha = 0$ corresponding to $\mathbf{a}_0$ and $\alpha = 1$ corresponding to $=a_1$. The term $r[\mathbf{p}; \mathbf{a}_0\, \mathbf{a}_1] = \alpha/(1-\alpha)$ is called the ratio of $\mathbf{p}$ with respect to $\mathbf{a}_0\, \mathbf{a}_1$.

Consider three points $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, and the affine combinations

$$\mathbf{b}_0 = (1-\alpha)\mathbf{a}_0 + \alpha\, \mathbf{a}_1 \quad \text{and} \quad \mathbf{b}_1 = (1-\alpha)\mathbf{a}_1 + \alpha\, \mathbf{a}_2,$$

both related by the same α, and the subsequent affine combination

$$\begin{aligned} \mathbf{p} &= (1-\beta)\mathbf{b}_0 + \beta\mathbf{b}_1 \\ &= (1-\alpha)(1-\beta)\mathbf{a}_0 + (\alpha(1-\beta) + (1-\alpha)\beta)\mathbf{a}_1 + \alpha\beta\mathbf{a}_2. \end{aligned} \tag{3}$$

Obviously, the resulting point $\mathbf{p}$ is symmetric in α and β, meaning that α and β can be interchanged. This symmetric configuration is referred to as A-frame lemma and is a fundamental tool in de Casteljau's work [16].

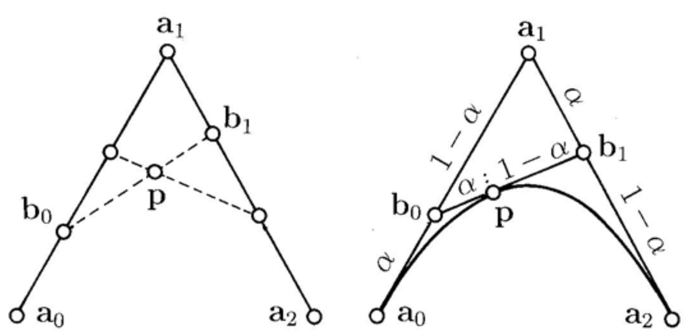

Figure 4: A-frame lemma and affine A-frame.

Let $\alpha = \beta$. Then (3) reduces to

$$\mathbf{p} = (1-\alpha)^2\mathbf{a}_0 + 2\alpha(1-\alpha)\mathbf{a}_1 + \alpha^2\mathbf{a}_2.$$

For fixed α the involved six points represent the so-called affine A-Frame, which is of great importance in Bernstein-Bézier methods. For varying α the point $\mathbf{p}$ traces out a parabola, defined by $\mathbf{a}_0$ and $\mathbf{a}_2$ with tangents that intersects in $\mathbf{a}_1$.

2 Conic Sections and Quadrics

The simplest figures in affine space besides lines and planes are conic sections, or more general, quadrics. They are studied conveniently by their quadratic equations.

2.1 Quadrics in Affine Space

In an affine space a quadric $\mathbf{Q}$ consists of all points $\mathbf{x}$ satisfying a quadratic equation

$$Q(\mathbf{x},\mathbf{x}) = \mathbf{x}^t C\mathbf{x} + 2\mathbf{c}^t\mathbf{x} + c = 0,$$

where $C = C^t$ is a symmetric non-zero matrix.

The intersection with a subspace is a quadric again. In particular, if the subspace is a line, one gets a pair of points. Note that these points can be real, coalescing or non-real.

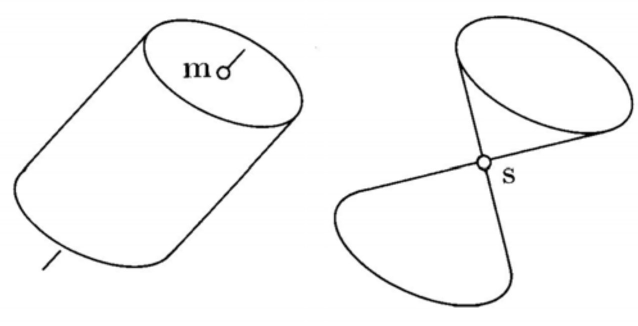

Figure 5: Midpoint and singular point.

A point $\mathbf{m}$ is called a midpoint of $\mathbf{Q}$ if $Q(\mathbf{x},\mathbf{x})$ is symmetric with respect to $\mathbf{m}$. This is the case for all solutions of

$$C\,\mathbf{m} + \mathbf{c} = \mathbf{o}.$$

Note that a solution may not exist. If a midpoint $\mathbf{s}$ lies on $\mathbf{Q}$, it satisfies

$$C\,\mathbf{s} + \mathbf{c} = 0\,, \quad \text{and} \quad \mathbf{c}^t\mathbf{s} + c = 0,$$

and is called a singular point, while $\mathbf{Q}$ degenerates to a cone.

2.2 Tangents and Polar Planes

A line **L**, given by $\mathbf{x} = \mathbf{q} + \lambda\,\mathbf{v}$, where **q** is a point of **Q**, intersects **Q** in a second point. If both points coalesce, then **L** is a tangent of **Q** at **q** and satisfies

$$[C\mathbf{q} + \mathbf{c}]^t \mathbf{v} = 0.$$

If additionally $\mathbf{v}^t C \mathbf{v} = 0$, then **L** lies completely on **Q**, and is called a generatrix of **Q**. Let $\mathbf{v} = \mathbf{q} - \mathbf{x}$, then **L** is a tangent if

$$Q(\mathbf{q}, \mathbf{x}) = [C\mathbf{q} + \mathbf{c}]^t \mathbf{x} + \mathbf{c}^t \mathbf{q} + c = 0.$$

This equation for **x** represents a plane, the tangent plane of **Q** at **q**.

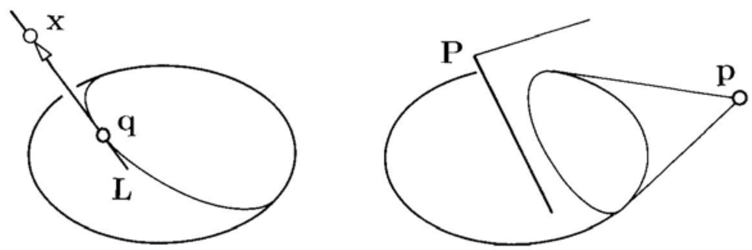

Figure 6: Tangent and polarity.

Replacing **q** by an arbitrary point **p** gives

$$Q(\mathbf{p}, \mathbf{x}) = \mathbf{p}^t C \mathbf{x} + \mathbf{c}^t \, [\mathbf{x} + \mathbf{p}] + c = 0.$$

It represents the polar plane **P** of the pole **p** with respect to **Q**. It intersects **Q** in points **q** with tangent planes through **p**. Note that these points **q** need not be real. Note also that $Q(\mathbf{p}, \mathbf{x})$ is symmetric in **p** and **x**.

A pair of points $\mathbf{p}, \mathbf{x}$ is called conjugate with respect to **Q** if $Q(\mathbf{p}, \mathbf{x}) = 0$. Hence the points of **Q** are self-conjugate with respect to **Q**. A pair of directions $\mathbf{u}, \mathbf{v}$ is called conjugate with respect to **Q** if $\mathbf{u}^t C \mathbf{v} = 0$. Conjugate elements play an important role when investigating quadrics in affine space.

Quadrics differ by the dimension of their midpoints or singularities, the dimension of their real generatrices and – in affine space – by the shape of their extensions to infinity.

2.3 Pascal's and Brianchon's Theorems

Conic sections have been studied extensively for several centuries. Of particular interest are the following two theorems on conic sectioncs in the plane:

> The three pairs of opposite sides of a hexagon inscribed to a conic section meet in three points of a line (*Pascal's theorem*).
>
> The three connections of opposite points of a hexilateral circumscribed to a conic section intersects in one point (*Brianchon's theorem*).

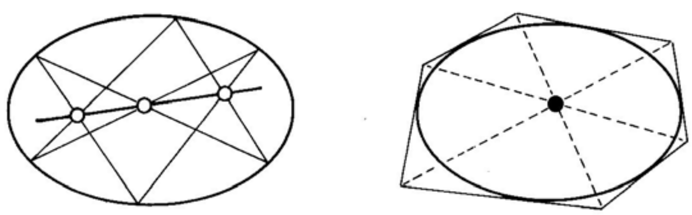

Figure 7: Pascal's and Brianchon's theorems.

As a consequence of these theorems, a conic section is uniquely determined by five points or five tangents in the plane.

Both theorems are of particular interest if pairs of consecutive points or tangents coalesce. E.g., let $\mathbf{a}_0$, $\mathbf{a}_2$ denote two points of a conic section with tangents meeting at a point $\mathbf{a}_1$. Let the points

$$\mathbf{b}_0 = (1-\alpha)\mathbf{a}_0 + \alpha\,\mathbf{a}_1, \quad \text{and} \quad \mathbf{b}_1 = \beta\,\mathbf{a}_1 + (1-\beta)\mathbf{a}_2. \tag{4}$$

span a third tangent. Its point of contact $\mathbf{p}$ is easily obtained from Brianchon's theorem,

$$\mathbf{p} = [\beta \mathbf{b}_0 + \alpha \mathbf{b}_1]/(\alpha+\beta),$$

where α and β as in (4), see also [6].

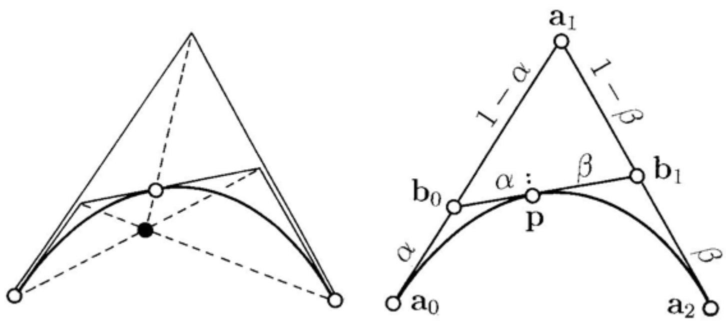

Figure 8: Affine representation of the projective A-frame.

3 Euclidean Space

The affine space $\mathbf{A}^n$ is a Euclidean space denoted by $\mathbf{E}^n$ and the corresponding vector space $\mathbf{V}^n$ is a Euclidean vector space if a dot product $< \mathbf{a}\,\mathbf{b} >= \mathbf{a}^t\mathbf{b}$ is give= n.

3.1 Cartesian Coordinates

An affine system $[\mathbf{a}_0, \mathbf{v}_1 \ldots \mathbf{v}_n]$ of $\mathbf{E}^n$ is called Cartesian if[3] $< \mathbf{v}_i\,\mathbf{v}_j >= \delta_{i,j}$ and it is positively oriented if $det[\mathbf{v}_1 \ldots \mathbf{v}_n] > 0$.

In Cartesian coordinates the distance of two points $\mathbf{p}$ and $\mathbf{q}$ is given by the length $\|\mathbf{v}\|$ of the vector $\mathbf{v} = \mathbf{q} - \mathbf{p}$,

$$dist(\mathbf{p}\,\mathbf{q}) = \|\mathbf{v}\| = \sqrt{\mathbf{v}^t\,\mathbf{v}},$$

and the angle φ of two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u}^t\,\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\,cos\varphi.$$

In particular, both vectors are called orthogonal if $cos\varphi = 0$, i.e. if $\mathbf{u}^t\,\mathbf{v} = 0$.

3.2 Gram-Schmidt Orthogonalization

A Cartesian system $[\mathbf{a}_0, \mathbf{b}_1 \ldots \mathbf{b}_d]$ of a subspace or Euclidean space itself can easily be constructed from an affine system $[\mathbf{a}_0,\, \mathbf{v}_1 \cdots \mathbf{v}_d]$ in $\mathbf{E}^n$ using the

[3] see footnote 2

Gram-Schmidt orthogonalization by alternating computation of the coefficients $\lambda_{i,j}$ and μ_i as follows:

Set $\mathbf{b}_1 = \mu_1 \mathbf{v}_1$ such that $\|\mathbf{b}_1\| = 1$.
Set $\mathbf{b}_2 = \mu_2(\mathbf{v}_2 + \lambda_{2,1}\mathbf{b}_1)$ such that $\mathbf{b}_2$ is orthogonal to $\mathbf{b}_1$ and $\|\mathbf{b}_2\| = 1.=$
...
Set $\mathbf{b}_d = \mu_d(\mathbf{v}_d + \lambda_{d,1}\mathbf{b}_1 + \cdots + \lambda_{d,d-1}\mathbf{b}_{d-1})$,
such that $\mathbf{b}_d$ is orthogonal to $\mathbf{b}_1, \ldots, \mathbf{b}_{d-1}$ and $\|\mathbf{b}_d\| = 1$.

Note that in a Cartesian system the dot product is written as $< \mathbf{u}\,\mathbf{v} >= \mathbf{u}^t\mathbf{v}$.

3.3 Euclidean Motions and Orthogonal Projections

If the frame $[\mathbf{a}_0, \mathbf{v}_1 \ldots \mathbf{v}_n]$ is Cartesian, then (1) represents a Cartesian coordinate transformation or a Euclidean motion. Simple examples of motions in 3-space are the translation by $\mathbf{v}$ and the rotation around the 3-axis by an angle ζ, in matrices written as

$$\mathbf{p} = \mathbf{v} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{p} = \mathbf{o} + \begin{bmatrix} \cos\zeta & -\sin\zeta & 0 \\ \sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x},$$

respectively. In particular, let $\mathbf{p} = B_i(\varphi)\,\mathbf{x}$ describe the rotation around the

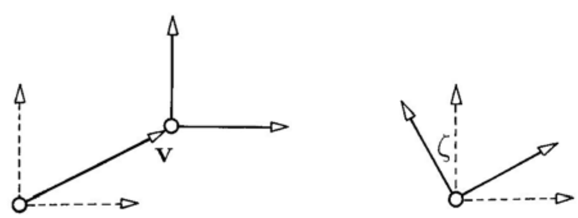

Figure 9: Translation and rotation.

i-axis by some angle φ. Any motion in 3-space can be written as

$$\mathbf{p} = \mathbf{v} + B_3(\gamma)B_1(\beta)B_3(\alpha)\,\mathbf{x},$$

where α, β, γ are the so-called Eulerian angles.

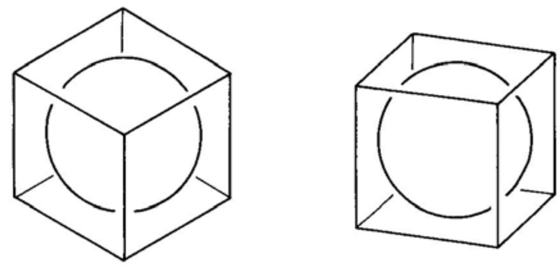

Figure 10: Isometric and dimetric orthogonal projection.

Any Euclidean motion followed by a map setting the coordinates $\eta_{d+1}, \ldots, \eta_n$ of the image $\mathbf{p}$ to zero results in an orthogonal projection onto some d-dimensional subspace .

It should be mentioned that orthogonal projections are more informative than simple parallel projections and much more informative than perspectivities. They are the only projections that map spheres to circles. Therefore orthogonal projections should be preferred in presenting technical objects.

3.4 Quadrics in Euclidean Space

If the vectors $\mathbf{v}_i$ of the Cartesian system $[\mathbf{a}_0, \mathbf{v}_1 \ldots \mathbf{v}_n]$ are pairwise conjugate with respect to a quadric $\mathbf{Q}$, then the $\mathbf{v}_i$ are principal axis directions of $\mathbf{Q}$ and C is a diagonal matrix.

One easily checks that for a conic section given by its equation a rotation by an angle α where

$$\tan 2\alpha = 2c_{12}/(c_{11} - c_{22})$$

turns the coordinate axes into the axis directions of $\mathbf{Q}$ and transforms C into diagonal form.

4 Projective Fundamentals

Introducing points at infinity leads to projective space and allows a unified and most elegant treatment of geometry[4].

[4] "All geometry is projective geometry" [Arthur Cayley 1821-1895]

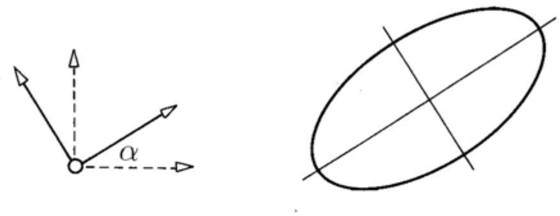

Figure 11: Principal axis transformation.

4.1 Homogeneous coordinates

Let $\xi_1, \ldots, \xi_n$ be affine coordinates of a point in $\mathbf{A}^n$ with respect to an affine frame $[\mathbf{a}_0, \mathbf{v}_1 \ldots \mathbf{v}_n]$ as above. Set $\xi_i = \beta_i / \beta_0$ with some $\beta_0 \neq 0$. Then the β_0,$\beta_1, \ldots, \beta_n$ are homogeneous coordinates with respect to the given affine frame. Note that any non-zero multiple of the homogeneous coordinate column $\mathbb{b} = [\beta_0 \beta_1 \ldots \beta_n]^t$ represents the same point. Note also that a point $\mathbb{p} = \mathbb{o}$ is undefined. It represents the so-called forbidden point.

As before $\beta_i = 0, i \neq 0$ represents the coordinate hyperplane $\xi_i = 0$. Further, $\beta_0 = 0$ represents points at infinity lying in the infinite or ideal hyperplane $\beta_0 = 0$. An affine space $\mathbf{A}^n$ together with its ideal hyperplane forms a projective space $\mathbf{P}^n$, the projective extension of $\mathbf{A}^n$.

The advantage of this extension is the symmetry of homogeneous coordinates. Points at infinity are handled as points in any other plane. In particular, ideal points allow to intersect parallel lines and subspaces – at infinity. Note that any non-zero multiple of a vector represents the same point at infinity.

Note also that $\beta_0 = 0$ and $\beta_0 \neq 0$ correspond to $\epsilon = 0$ and $\epsilon = 1$ in Subsection **1.2** above.

4.2 Projective Coordinates

Let $\mathbb{a}_0, \ldots, \mathbb{a}_d$ be linearly independent columns of homogeneous coordinates of $d+1$ points in $\mathbf{P}^n$ with integrated factors such that the sum $\mathbb{a} = \mathbb{a}_0 + \cdots + \mathbb{a}_d$ represents a given further point $\mathbb{a}$ called the unit point. These $d+2$ points determine a projective frame $[\mathbb{a}_0, \ldots, \mathbb{a}_d \,; \mathbb{a}]$ of some projective subspace $\mathbf{S}$ spanned by the points $\mathbb{a}_0, \ldots, \mathbb{a}_d$. Any point $\mathbb{p}$ of this subspace can be represented by homogeneous projective coordinates $\mathbb{x} = [\xi_0 \dots \xi_d]^t$ as

$$\rho\, \mathbb{p} \;= \xi_0 \mathbb{a}_0 + \xi_1 \mathbb{a}_1 + \cdots + \xi_d \mathbb{a}_d, \quad \rho \neq 0. \tag{5}$$

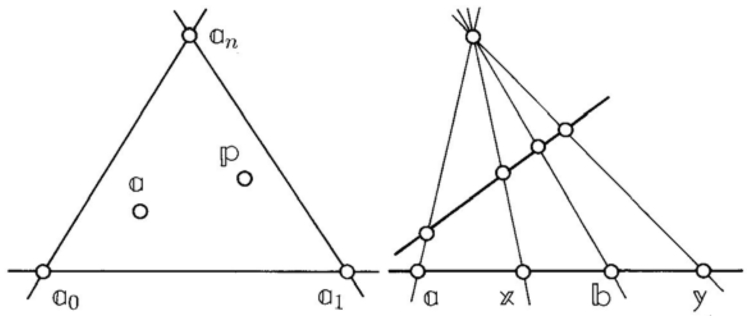

Figure 12: Projective system and cross ratio.

In particular, if $\mathbb{a}_i = [1, \mathbf{a}_i^t]^t$, and $\mathbb{a} = [1, \mathbf{a}^t]^t$ then $\mathbf{a}$ is the center $[\mathbf{a}_0 + \dots + \mathbf{a}_d]/d$ of the $\mathbf{a}_i$ and the ξ_i are a multiple of the barycentric coordinates of $\mathbf{p}$ with respect to the affine frame $[\mathbf{a}_0 \dots \mathbf{a}_d]$.

4.3 Projective Maps

The representation (5), with matrices written as $\rho\mathbb{p} = \mathbb{A}\mathbb{x}$, allows two interpretations. First it represents the point $\mathbb{p} \in \mathbf{S}$ by new homogeneous coordinates $\mathbb{x}$. Second it represents a projective map $\psi : \mathbb{x} \to \mathbb{p}$ of $\mathbf{S}$ into $\mathbf{P}^n$.

In particular, ψ maps the fundamental points $\mathbb{x}_i = [\delta_{0,i} \dots \delta_{d,i}]^t$ into $\mathbb{a}_i$, $i = 0, \dots, d$ and the unit point $[1 \dots 1]^t$ into $\mathbb{a}$. This determines the projective map uniquely - and $\mathbb{A}$ except for a common factor ρ.

Note that a projective map does not preserve parallelism and ratio in general, but it preserves the cross ratio

$$cr[\mathbb{x}\mathbb{y}; \mathbb{a}\mathbb{b}] = r[\mathbf{x}; \mathbf{ab}]/r[\mathbf{y}; \mathbf{ab}].$$

In particular, if $cr[\mathbb{x}\mathbb{y}; \mathbb{a}\mathbb{b}] = -1$ then both pairs of points, $\mathbb{x}\,\mathbb{y}$ and $\mathbb{a}\,\mathbb{b}$, are in harmonic position. For example, let α be an affine scale, see Subsection **1.6**, the pairs of points corresponding to $-1, +1$ and $0, \infty$ are in harmonic position.

4.4 The Procedure of Inhomogeneizing

Any homogeneous equation in projective coordinates can easily be inhomogeneized by setting the homogeneizing coordinate to one. Any point $\mathbb{x} = [\xi_0, \xi_0 \mathbf{x}^t]^t$ or $\mathbb{v} = [0, \mathbf{v}^t]^t$ is simply inhomogeneized to $\mathbf{x}$ or $\mathbf{v}$, respectively.

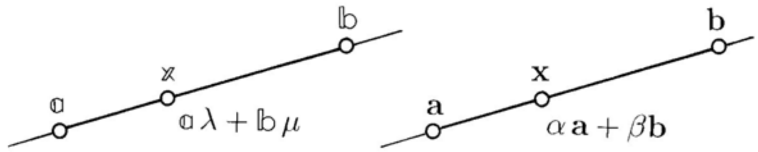

Figure 13: Inhomogeneizing the point of a line.

Of particular interest is the application of this procedure to the point $\mathbb{x}$ of a projective line given by

$$\rho\, \mathbb{x} = \lambda\, \mathbb{a} + \mu\, \mathbb{b}.$$

Let $\rho = 1$ and let $\mathbb{a} = [\alpha_0, \alpha_0 \mathbf{a}^t]^t$ and $\mathbb{b} = [\beta_0, \beta_0 \mathbf{b}^t]^t$. Then inhomogeneizing $\mathbb{x} = [\xi_0, \xi_0 \mathbf{x}^t]^t$ results in the affine combination

$$\mathbf{x} = \alpha\, \mathbf{a} + \beta\, \mathbf{b}, \quad \text{where} \quad \alpha = \lambda\, \alpha_0/\xi_0 \quad \text{and} \quad \beta = \mu\, \beta_0/\xi_0. \tag{6}$$

Similar results are obtained by inhomogeneizing the points of a projective subspace (5) of higher dimension.

4.5 Repeated Projective Combinations

Repeated affine combinations and A-frames are often used in CAGD to compute polynomial curves and surfaces and can also be applied to the homogeneous coordinates of rational curves and surfaces. It is useful to inhomogeneize the resulting projective combinations by the procedure demonstrated above.

Moreover, after an initial inhomogeneizing one can continue with the affine representation of the projective A-frame presented in Subsection **2.3**. For more details on this procedure and its applications see [6].

4.6 Quadrics in Projective Space

In homogeneous or projective coordinates $\mathbb{x}$ the equation of a quadric $\mathbf{Q}$ simplifies to

$$Q(\mathbb{x}\,\mathbb{x}) = \mathbb{x}^t\,\mathbb{C}\,\mathbb{x} = 0, \quad \text{where} \quad \mathbb{C}^t = \mathbb{C},$$

and the polarity is written as

$$Q(\mathbb{p}\,\mathbb{x}) = \mathbb{p}^t\,\mathbb{C}\,\mathbb{x} = 0.$$

Note that in homogeneous coordinates the midpoint of $\mathbf{Q}$ is the pole of the infinite plane. Note also that $\mathbf{Q}$ is a cylinder, if it has a singular point at infinity, etc.

4.7 Parametrizing a Quadric and Its Equation

If a quadric $\mathbf{Q}$ is given by its equation $Q(\mathbb{x}\,\mathbb{x}) = 0$, and $\mathbb{q}$ represents a point of $\mathbf{Q}$, i.e., $Q(\mathbb{q}\,\mathbb{q}) = 0$, then

$$\mathbb{x}\,\rho = Q(\mathbb{p}\,\mathbb{p})\;\mathbb{q} - 2\,Q(\mathbb{p}\,\mathbb{q})\;\mathbb{p}$$

is a parametrization of $\mathbf{Q}$, which is quadratic in the coordinates of $\mathbb{p}$. Setting, e.g., $\mathbb{p} = \mathbb{p}_0\zeta_0 + \cdots + \mathbb{p}_{n-1}\zeta_{n-1}$, where the $\mathbb{p}_i$ are suitably choosen, it is also quadratic in the homogeneous ζ_i. Note that one may use any other representation for $\mathbb{p}$, e.g., polar coordinates with the center $\mathbf{q}$.

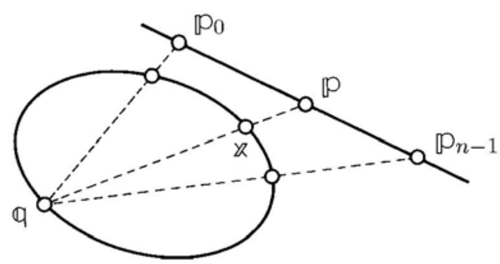

Figure 14: Parametrizing a quadric.

Conversely, the equation $Q(\mathbb{x}\,\mathbb{x}) = 0$ of a quadric in $\mathbf{A}^n$ or $\mathbf{P}^n$ depends on $r = (n+1)(n+2)/2$ homogeneous coefficients, the elements of $\mathbb{C}$. Let $r-1$ pairs of conjugate points, $\mathbb{p}_i$ and $\mathbb{q}_i$, be given, and let $\mathbb{x}$ denote some arbitrary point of **Q**. Then the $r-1$ conditions $Q(\mathbb{p}_i\,\mathbb{q}_i)=0$ together with $Q(\mathbb{x}\,\mathbb{x})=0$ form a homogeneous linear system for the r unknown coefficients of $\mathbb{C}$. Setting its determinant to zero results in the equation of **Q**.

5 Duality

In homogeneous or projective coordinates the equation of a hyperplane simplifies to

$$\mathbb{u}^t\mathbb{x} = u_0\xi_0 + u_1\xi_1 + \cdots + u_n\xi_n.$$

The u_i are homogeneous coordinates of the hyperplane – just as the ξ_i for $\mathbb{x}$. Homogeneous coordinates can either represent a point or a hyperplane. Consequently any configuration of points and hyperplanes has a dual configuration of hyperplanes and points, where the dual of a point or hyperplane is a hyperplane or point represented by the same coordinates. More generally, the hyperplanes containing some points $\mathbb{b}_1,\ldots,\mathbb{b} =_d$ are dual to the points of intersection of the hyperplanes $\mathbb{b}_1,\ldots,\mathbb{b}_d$, and vice versa.

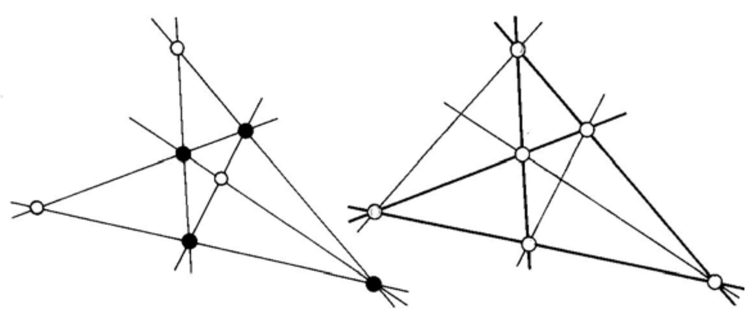

Figure 15: Quadrangle and dual quadrilateral.

Note that the duality depends on the dimension of the space. For example, Pascal's and Brianchon's configuration are dual in the plane, where points and tangents of a conic section are dual elements.

6 Osculating Curves and Surfaces

An important task in CAGD is to connect curves and surfaces smoothly.

6.1 Curve and Surface

A curve $\mathbf{x}(t)$ in affine space $\mathbf{A}^n$ is called regular at t_0 if $\dot{\mathbf{x}}(t_0) \neq \mathbf{o}$.

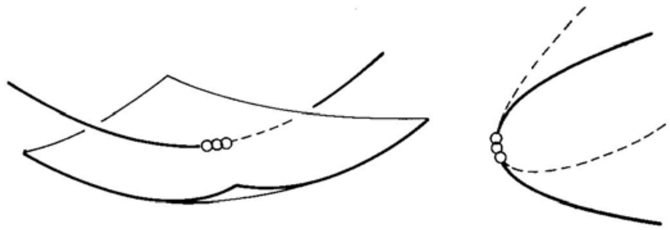

Figure 16: Contact of order two.

The curve $\mathbf{x}(t)$ is said to have a contact of order r at t_0 with a surface given by the equation $F(\mathbf{x}) = 0=$, if it is regular at t_0 and if $F(\mathbf{x}(t))$ and its derivatives up to order r vanish at $t = t_0$. This means geometrically that the curve has $r + 1$ coalescing points in common with the surface at $t = t_0$. Note that this definition does not depend on the parametrization of $\mathbf{x}(t)$. Note also that by its geometric meaning the contact of order r is projectively invariant.

6.2 Curve and Curve

A second curve $\mathbf{y}(s)$ given by the intersection of a number of surfaces contacts $\mathbf{x}(t)$ at t_0 with order r if $\mathbf{x}(t)$ contacts all these surfaces with at least order r.

If the second curve $\mathbf{y}(s)$ is given parametrically, then both curves have contact of order r at t_0 if there exists a regular reparametrization $t = t(s)$ for $\mathbf{x}(t)$ such that the Taylor expansions of $\mathbf{x}(t(s))$ and $\mathbf{y}(s)$ agree at $t_0 = t(s_0)$ up to order r. This condition can be expressed by the chain rule as a system of $r + 1$ linear equations. Therefore contact of order r is referred to as chain rule continuity.

For example, a curve and its osculating circle at a point t_0 have contact of order 2.

6.3 Surface and Surface

Two surfaces have contact of order r at $\mathbf{p}$ if all regular curves that lie on one of them and meet $\mathbf{p}$ have at least contact of order r with the other surface. This means that after a suitable reparametrization the Taylor expansions of both surfaces at $\mathbf{p}$ agree up to order r.

6.4 Contour Lines, Reflection Lines and Isophotes

There are some important helpful curves to check the smoothness of surfaces visually.

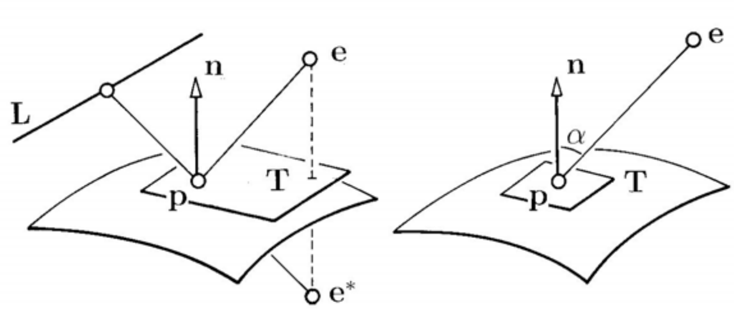

Figure 17: Reflection line and isophote.

> A reflection line on a surface consists of all points $\mathbf{p}$ whose connection with some fixed point $\mathbf{e}$, the eye, is reflected into a ray that meets a given fixed line $\mathbf{L}$.
>
> An isophote on a surface consists of all points $\mathbf{p}$ whose connection with the eye $\mathbf{e}$ forms a fixed angle with the surface normal at $\mathbf{p}$.
>
> Contour lines are special isophotes. They consist of all points $\mathbf{p}$ where the tangent plane meets $\mathbf{e}$.

In general, on composite surfaces contur lines, reflection lines and isophotes have a lower order of contact than the surfaces themselves.

Note that all three kinds of curves may consist of several parts. Note also that the use of infinite elements $\mathbf{e}$ and $\mathbf{L}$ simplifies their computation.

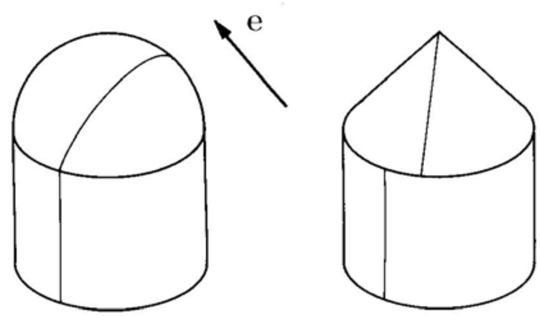

Figure 18: Contour lines.

7 Differential Fundamentals

Arc length, curvature and torsion describe the local properties of a curve, the curvature of so-called principal normal sections describe the local properties of surfaces. The main tool for such investigations is a local frame.

7.1 Arc Length and Osculating Plane

Let a curve in $\mathbf{E}^3$ be given parametrically as $\mathbf{x} = \mathbf{x}(t)$ and let

$$\mathbf{a}_0 = \mathbf{x}(t_0), \quad \mathbf{v}_1 = \dot{\mathbf{x}}(t_0), \quad \mathbf{v}_2 = \ddot{\mathbf{x}}(t_0),$$

denote its point and first two derivatives at some t_0. If $\mathbf{v}_1 \neq 0$, its tangent at $\mathbf{a}_0$ is given by

$$\mathbf{p} = \mathbf{a}_0 + \xi_1 \mathbf{v}_1.$$

If $\mathbf{v}_1 \wedge \mathbf{v}_2 \neq \mathbf{o}$, its osculating plane at $\mathbf{a}_0$ is given by

$$\mathbf{p} = \mathbf{a}_0 + \xi_1 \mathbf{v}_1 + \xi_2 \mathbf{v}_2.$$

The differential term $ds = \|\dot{\mathbf{x}}(t)\| dt$ is called the arc element of $\mathbf{x}(t)$, and the integral

$$s(t) = \int_{t_0}^{t} \|\dot{\mathbf{x}}(t)\| dt$$

its arc length, beginning at t_0. The arc length represents the natural parameter of the curve. In most cases it can only be computed approximately.

7.2 Curvature and Torsion

The natural parameter s is very helpful to derive general curve properties. E.g., let $\alpha(s)$ and $\beta(s)$ denote the angles of the tangent and the osculating plane at s with the tangent and osculating plane at some s_0, respectively, and let the prime $'$ denote differentiation with respect to arc length. Then

$$\kappa = \alpha'(s_0) \quad \text{and} \quad \tau = \beta'(s_0)$$

are called curvature and torsion of $\mathbf{x}(t)$ at s_0, respectively. Note that $\rho = 1/\kappa$ represents the radius of the osculating circle.

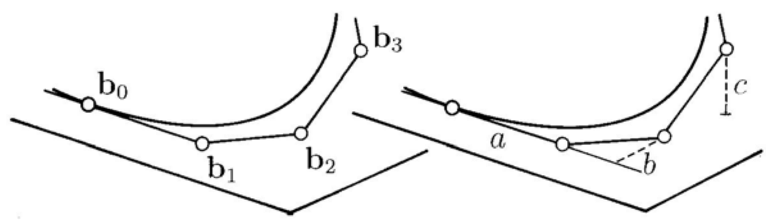

Figure 19: Curvature and torsion of a rational Bézier curve.

For example, a rational curve of degree n with Bézier points $\mathbb{b}_i = [\beta_i, \beta_i \mathbf{b}_i^t]^t$ has the span of $\mathbf{b}_0$, $\mathbf{b}_1$ as tangent at $\mathbf{b}_0$, and the span of $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$ as osculating plane at $\mathbf{b}_0$. At $t = 0$ one has

$$\kappa(0) = \frac{n-1}{n} \frac{\beta_0 \beta_2}{\beta_1^2} \frac{b}{a^2} \quad \text{and} \quad \tau(0) = \frac{n-2}{n} \frac{\beta_0 \beta_3}{\beta_1 \beta_2} \frac{c}{a\,b},$$

where a, b, c denote the distances of $\mathbf{b}_1$ from $\mathbf{b}_0$, of $\mathbf{b}_2$ from the tangent at $\mathbf{b}_0$, and of $\mathbf{b}_3$ from the osculating plane at $\mathbf{b}_0$, respectively.

7.3 The Frenet Frame

Gram-Schmidt orthogonalization applied to $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$, $\dddot{\mathbf{x}}$ at $\mathbf{x}(t_0)$ results in the so-called Frenet frame $[\mathbf{t}\,\mathbf{m}\,\mathbf{b}]$, which depends on t. One has

$$[\mathbf{t}'\,\mathbf{m}'\,\mathbf{b}'] = [\mathbf{t}\,\mathbf{m}\,\mathbf{b}] \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix},$$

which is an important tool for further investigations, see [3], [5] and [9].

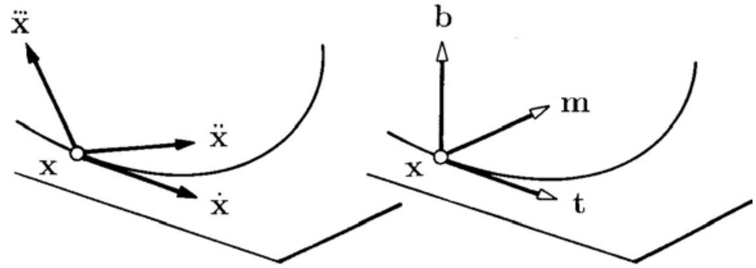

Figure 20: Curve with derivatives, osculating plane and Frenet frame.

7.4 Curves on Surfaces

Let a surface be given parametrically as

$$\mathbf{x} = \mathbf{x}(u, v) = \mathbf{x}(\mathbf{u}).$$

The lines $u = fixed$, $v = fixed$ are called iso-lines. If the partial derivatives $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent, the surface normal is defined by

$$\mathbf{n} = [\mathbf{x}_u \wedge \mathbf{x}_v]/\|\mathbf{x}_u \wedge \mathbf{x}_v\|.$$

Let $\mathbf{u} = \mathbf{u}(t)$ denote some curve in the $\mathbf{u}$-plane. Then, in general, $\mathbf{x} = \mathbf{x}(\mathbf{u}(t))$ represents a curve on the surface. The arc element ds of this curve is given by its square

$$ds^2 = (E\,\dot{u}^2 + 2\,F\,\dot{u}\,\dot{v} + G\,\dot{v}^2)\,dt^2,$$

where $E = \mathbf{x}_u^t\mathbf{x}_u$, $F = \mathbf{x}_u^t\mathbf{x}_v$, and $G = \mathbf{x}_v^t\mathbf{x}_v =$ are well-known as Gaussian first fundamental quantities. Note that $\|\mathbf{x}_u \wedge \mathbf{x}_v\|^2 = E\,G - F^2$.

7.5 Meusnier's Sphere and Dupin's Indicatrix

Consider all curves on a surface meeting a given point $\mathbf{p}$ with a given tangent $\mathbf{t}$ there. One has :

> The osculating circles of these curves lie on a sphere (*Meusnier's sphere*).

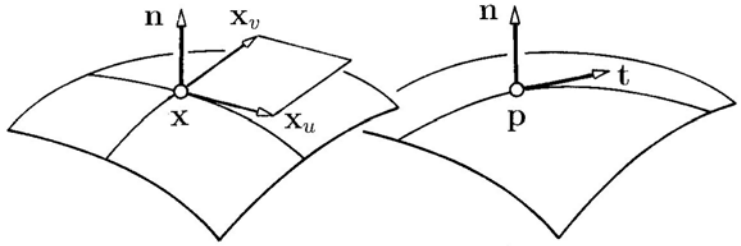

Figure 21: Local frames on a surface.

It follows that the radius of the osculating circle of a surface curve is given by $\rho = \rho_0 \, cos \, \delta$, where δ denotes the angle between the surface normal and the osculating plane and ρ_0 is the radius of Meusnier's sphere. The inverse $\kappa_0 = 1/\rho_0$ is called the normal curvature of the surface at $\mathbf{p}$ in direction of $\mathbf{t}$. Hence it is sufficient to know the curvature of one of these curves and the angle of its osculating plane with $\mathbf{n}$ in order to compute all others= =2E

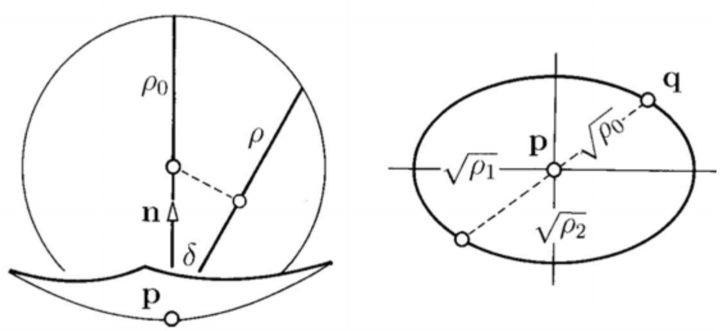

Figure 22: Meusnieur's sphere and Dupin's indicatrix.

In general, the normal curvature κ_0 differs for different $\mathbf{t}$. For all tangent directions $\mathbf{t}$ at $\mathbf{p}$ consider the points

$$\mathbf{q} = \mathbf{p} + \sqrt{\rho_0}\,\mathbf{t}$$

of the tangent plane with distance $\sqrt{\rho_0}$ from $\mathbf{p}$. One has:

> The points $\mathbf{q}$ lie on a conic section with center $\mathbf{p}$ (*Euler's theorem*).

This conic section is also known as Dupin's indicatrix. Its axis directions are called the principal directions, and the corresponding values of $\kappa = 1/\rho_0$ are called the principa= l curvatures of the surface at $\mathbf{p}$, mostly denoted by $\kappa_1 = 1/\rho_1$ and $\kappa_2 = 1/\rho_2$.

Note that Dupin's indicatrix can be an ellipse, a pair of parallel lines, or a hyperbola. In case of a hyperbola it has two real asymptotic directions . The normal curvature is zero there and ρ_0 is infinite.

If $\kappa_1 = \kappa_2$, then Dupin's indicatrix is a circle and $\mathbf{p}$ is called an umbilical or spherical point.

7.6 The Curvatures of a Surface

Because of its geometric meaning Dupin's indicatrix and consequently the principal curvatures κ_1 and κ_2 at a point $\mathbf{p}$ do not depend on the parametric representation of the surface.

The expressions

$$K = \kappa_1 \kappa_2 \quad \text{and} \quad H = (\kappa_1 + \kappa_2) = /2$$

are called Gaussian curvature and mean curvature of the surface at $\mathbf{p}$, respectively. Both give important information about the smoothness of a surface. Moreover, Gauss has shown that K depends on E, F, and G and their derivatives only. This means that K depends on the inner measurements on the surface only and is invariant under deformations of the surface that do not distort the measurement of lengths on the surface.

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