TRIANGULAR G^k-SPLINES*

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Abstract. In this paper a new approach is presented to construct piecewise polynomial G^k-surfaces of arbitrary topology and smoothness order $k \geq 1$ of degree $\mathcal{O}(k)$. This approach generalizes some results presented in 1997 in CAGD and in 1999 at the Saint-Malo conference, respectively.

In our construction only $4n$ polynomial patches are needed to fill an n-sided hole in a generalized C^k-(half-)box spline surface. This is achieved by coalescing certain control points while at the same time maintaining a regular parametrization.

1. Introduction. The problem in modelling smooth surfaces of arbitrary topology arises from surfaces that cannot be represented in box spline form. To this category belong e.g. all surfaces of topological genus not equal to one. The reason for this restriction is that a surface can only be represented in box spline form if it can be approximated by a regular quadrilateral or triangular control net of the same topological type. Thereby the regularity constraint means that exactly four or six meshes must meet at every interior control point, respectively. Thus for arbitrary surfaces nets including control points that do not obey this constraint must be used. These control points are usually called *irregular vertices of valence* n, where n denotes the number of meshes meeting in that particular control point.

In an arbitrary quadrilateral or triangular control net the irregular vertices are the centres of subnets that do not correspond to box spline surfaces. So the *generalized* box spline surface, which is defined by all regular subnets, has n-sided holes corresponding to irregular vertices of valence n.

Constructing smooth fillings for those holes is known as the *n-sided hole problem*, which has been addressed by many authors.

One approach is based on generalized subdivision schemes, where the hole is filled iteratively with ever smaller surface rings, see e.g. [3, 5, 6, 12]. Although it can be proved that the resulting surfaces are smooth, they consist of infinitely many patches. So far no subdivision algorithms are known to generate G^k-surfaces of arbitrary smoothness order $k \geq 1$.

A second major approach is to explicitely construct a finite number of polynomial patches that join smoothly to fill the hole. All works in this area utilize the concept of geometric continuity as introduced by [4, 8, 9, 10]. Most of these works are concerned with the construction of G^1- or G^2-surfaces, see e.g. [7, 14, 15, 16, 17, 19, 24]. The fillings consist of a finite number of polynomial patches of relatively low degree, namely ≥ 2 for G^1 and ≥ 5 for G^2. Because of the complicated smoothness constraints these constructions cannot be extended to arbitrary smoothness orders.

J. Hahn [11] proposed in 1989 a method to construct G-splines of arbitrary smoothness order, which later was refined by H. Mögerle [13]. This was achieved by fillings of relatively high polynomial degree of $\mathcal{O}(k^2)$.

The freeform splines of H. Prautzsch [18] overcome this problem for tensor-product B-spline surfaces via polynomial patches whose bidegree is only linear in k. This construction has lately been extended to triangular splines [23]. Thereby the fillings consist of $\mathcal{O}(k^2)$ many polynomial patches.

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*Supported by IWRMM and DFG grant # PR 565/1-1.

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A similar approach has been taken by U. Reif [20]. He uses only a (in k) constant number of patches which are singularly parametrized.

The above list of solutions of the n-sided hole problem is far from complete, but it may serve us to distinguish our approach from existing solutions. Our G^k-fillings for $k \geq 1$, that we will call in the sequel p(olynomial)-patches, consist of a constant number of triangular, polynomial patches. The degree of these patches is linear in k and the resulting surfaces are regularly parametrized.

In this paper we first describe the construction of G^k-p-patches based on bivariate, symmetric box spline over the three-directional grid, i.e. for even k (Section 2). Subsequently we extend this to odd smoothness orders k using the corresponding half-box splines (Section 3). In Section 4 we discuss special details of out method, namely how to construct planar, regular G^k-surface rings with coalescing central control points.

2. P-Patches for Box Splines. The bivariate, symmetric box splines over the three-directional grid shown in Figure 2.1 are recursively defined by

$$N_1(\mathbf{u}) = \text{the hexagonal pyramid over } V \cdot [0,1)^2,$$

$$N_m(\mathbf{u}) = \int_{[0,1)^3} N_{m-1}(\mathbf{u} - V \cdot \mathbf{t})\, d\mathbf{t}$$

for $m \geq 2$, where $V := [\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{t} = [t_1\, t_2\, t_3]^t$.

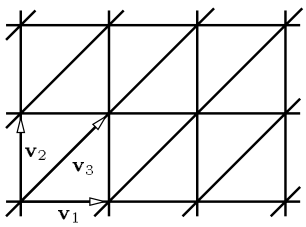

FIG. 2.1. *The three-directional grid.*

These box splines are k-times continuously differentiable with $k = 2(m-1)$ and over every triangle of the grid polynomial of degree $d = 3m - 2$. A single polynomial segment is called a *patch* and a surface made up of these patches a *box spline surface.* For details of box splines see [2].

Every patch of a box spline surface is determined by a regular triangular control net consisting of $3(k+2)^2/4$ control points arranged as in Figure 2.2.

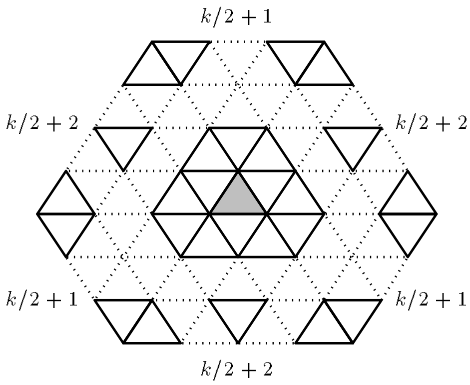

FIG. 2.2. *The regular triangular net of one patch (gray) of a box spline of degree* $d = 3k/2 + 1$.

For the modelling of arbitrary surfaces control nets including irregular vertices are necessary. In an arbitrary trinagular net the regular subnets of the form of Figure 2.2 still determine single box spline patches. The union of all these patches make up a *generalized* box spline surface, which has n-sided holes corresponding to the irregular vertices of valence n.

An example for a generalized box spline surface of degree 7 is shown in Figure 2.3: To the left is a control net with one vertex of valence 8 and to the right the corresponding generalized box spline surface with one 8-sided hole.

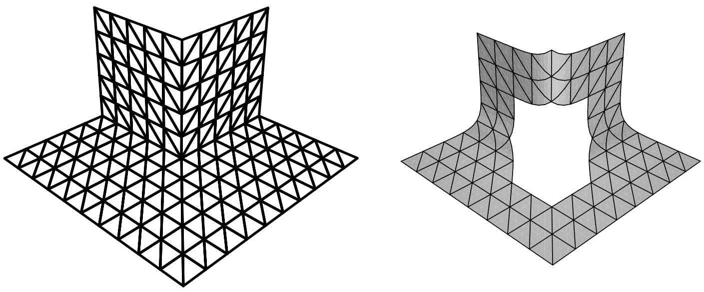

FIG. 2.3. *A triangular control net (left) with a vertex of valence 8 and the corresponding generalized box spline surface of degree 7 and its patches (right).*

Because of the size of the net in Figure 2.2 at least $k+1$ rings of regular vertices around one irregular vertex are necessary for a one-to-one correspondence between holes and irregular vertices. In this case every hole is surrounded by a complete box spline surface ring consisting of $(k+1)n$ patches, see Figure 2.3. In [23] we proposed to fill such a hole using k^2n patches. Instead we will describe here how a filling with $4n$ patches can be constructed.

Assume that the innermost $k/2-1$ rings of control points around an irregular vertex coalesce with the irregular vertex. Thus the irregular vertex becomes a *multiple vertex*. In every subnet, that has the form of Figure 2.4, the multiple vertex can be replaced by a suitable regular triangular net whose vertices are all equal to the multiple vertex. With this each of the n possible subnets determine $k(k+4)/4$ patches. Together these patches form $k/2-1$ box spline surface rings in the hole, whose innermost ring $\mathbf{r}$ consists of $3n$ patches, see Figure 2.4.

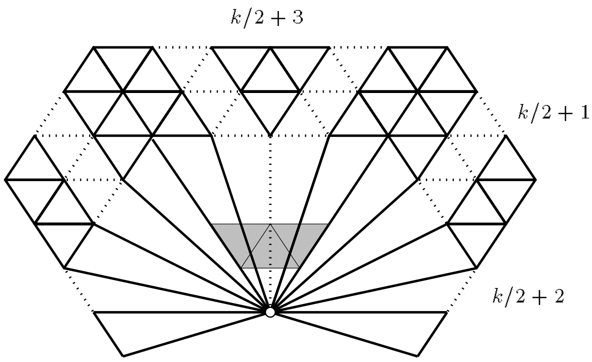

FIG. 2.4. *The control net of three patches (gray) of a box spline of degree $d = 3k/2+1$ with a multiple control point ($\circ$).*

Now this shrinked hole can be filled by the analogous p-patch construction of [23]:
CONSTRUCTION 2.1.

Input: *C^k-surface ring $\mathbf{r}$ consisting of $3n$ box spline patches of degree d bounding an n-sided hole.*

Output: *Regular G^k-surface of degree $2d$.*

1. Plane parametrization: *Construct $4n$ planar box spline patches $\mathbf{x}_1, \ldots, \mathbf{x}_{4n}$ of degree d with coalescing $k/2-1$ inner control point rings and C^k- and C^0-joints, respectively, as in Figure 2.5, such that the* plane parametrization $\mathbf{x} := \cup \mathbf{x}_i$ *is a regular and injective G^∞-surface.*

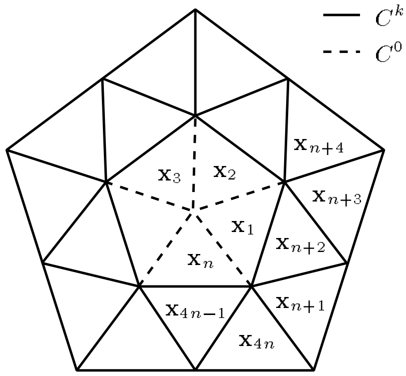

FIG. 2.5. *The $4n$ planar patches $\mathbf{x}_1, \ldots, \mathbf{x}_{4n}$ with its C^k- and C^0-joints for $n = 5$.*

2. Reparametrization: *Choose a quadratic polynomial $\mathbf{q}(u,v) : \mathbb{R}^2 \to \mathbb{R}^3$ and reparametrize it over $\mathbf{x}_1, \ldots, \mathbf{x}_{4n}$:*

$$\mathbf{p}_i(u,v) = \mathbf{q}(\mathbf{x}_i(u,v)), \qquad i = 1, \ldots, 4n.$$

The p(olynomial)-patch $\mathbf{p} := \cup \mathbf{p}_i$ *is a G^k-surface of degree $2d$.*

3. Adaptation: *Adapt the p-patch $\mathbf{p}$ and the boundary $\mathbf{r}$ to each other, e.g. by exchanging certain control points.*

In Construction 2.1 one question remains which will be analyzed in Section 4: Can a regular and injective plane parametrization with coalescing control points be constructed (Section 4)?

Before we deal with these details, we will extend the above Construction 2.1 to odd smoothness orders k using half-box splines.

3. P-Patches for Half-Box Splines. The bivariate, symmetric half-box splines over the three-directional grid shown in Figure 2.1 are recursively defined by (cf. [21])

$$H_1^{\triangle}(\mathbf{u}) = \begin{cases} 1 & \text{if } \mathbf{u} \in \boldsymbol{\triangle} \\ 0 & \text{else} \end{cases} \quad \text{resp.} \quad H_1^{\triangledown}(\mathbf{u}) = \begin{cases} 1 & \text{if } \mathbf{u} \in \boldsymbol{\triangledown} \\ 0 & \text{else} \end{cases},$$

$$H_m^{\triangle}(\mathbf{u}) = \int_{[0,1)^3} H_{m-1}^{\triangle}(\mathbf{u} - X \cdot \mathbf{t}) d\mathbf{t} \quad \text{resp.} \quad H_m^{\triangledown}(\mathbf{u}) = \int_{[0,1)^3} H_{m-1}^{\triangledown}(\mathbf{u} - X \cdot \mathbf{t}) d\mathbf{t}$$

for $m \geq 2$, where $\boldsymbol{\triangle} := \{(u,v) \,|\, 0 \leq v \leq u < 1\}$ and $\boldsymbol{\triangledown} := [0,1)^2 \setminus \boldsymbol{\triangle}$. These half-box splines are k-times continuously differentiable with $k = 2(m-1)-1$ and over every triangle of the grid polynomial of degree $d = 3(m-1)$. A surface made up of half-box spline patches is called a *half-box spline surface*.

A *regular hexagonal net* is the dual of a regular triangular net. Every patch of a half-box spline surface is determined by a regular hexagonal control net, which is dual to a triangular net in Figure 2.2, consisting of $(3k^2 + 12k + 11)/2$ control points.

For the modelling of arbitrary surfaces also arbitrary hexagonal control nets are necessary, which are the duals of arbitrary triangular nets. Thus an arbitrary hexagonal control net contains so-called *irregular meshes*, which are n-sided with $n \neq 6$. The union of all patches determined by the regular subnets of an arbitrary hexagonal control net make up a *generalized* half-box spline surface. This surface has n-sided holes corresponding to the irregular, n-sided meshes.

Due to the size of the control net of a single patch, there must be for a one-to-one correspondence of holes to irregular meshes at least $k+1$ rings of control points between two rings of control points, that belong to two irregular meshes. Under this assumption every hole is surrounded by a complete half-box spline surface ring consisting of $(k+2)n$ patches.

Analogous to the box spline case we will assume that the innermost $(k-1)/2$ rings of control points around an irregular mesh and the vertices of the irregular mesh coalesce in a multiple vertex. This defines $(k-1)/2$ additional surface rings in the hole, whose innermost ring $\mathbf{r}$ consist of $3n$ patches. For this surface ring $\mathbf{r}$ a p-patch can be constructed using Construction 2.1 if all involved patches are half-box spline patches.

REMARK 3.1. *For more details on p-patches for half-box splines see [22].*

4. The Plane Parametrization. To prove that a regular and injective plane parametrization $\mathbf{x}$ with coalescing inner control points can be constructed, we define a control net that determines such a plane parametrization. First we discuss with the box spline case and afterwards the half-box spline case.

Let $\mathcal{C}$ be a control net which consist of $k/2+2$ rings of control points around one irregular vertex of order n. The inner $k/2-1$ rings of control points coalesce with the irregular vertex. The control net $\mathcal{C}$ is generated by rotating the *net segment* $\mathcal{C}^1$ in Figure 4.1 around the origin by $2(i-1)\pi/n, i = 1, \ldots, n$.

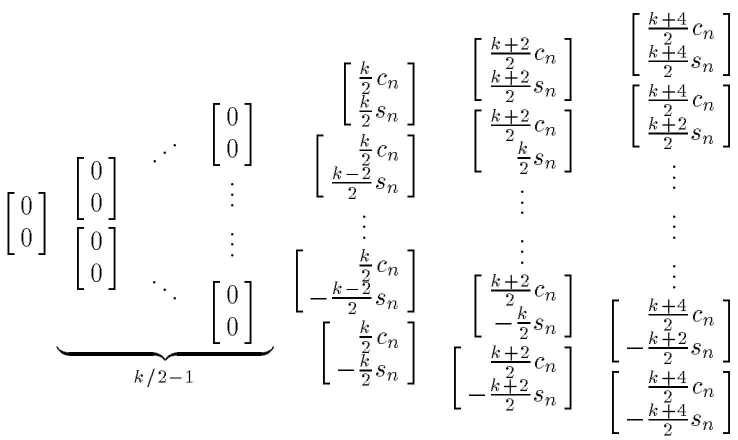

FIG. 4.1. *The net segment $\mathcal{C}^1$ with $c_n := \cos(\pi/n), s_n := \sin(\pi/n)$.*

Then $\mathcal{C}$ determines a generalized box spline surface ring with $3n$ patches of degree $d = 3k/2+1$. These patches are the patches $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$ of the wanted plane parametrization $\mathbf{x}$. Every regular subnet $\mathcal{X}^i$ of $\mathcal{C}$ of the form of Figure 2.4 determines a *segment* $\mathbf{x}^i := \mathbf{x}_{n+3i-2} \cup \mathbf{x}_{n+3i-1} \cup \mathbf{x}_{n+3i}$ for $i = 1, \ldots, n$. Because the segment

$\mathbf{x}^i$ is generated from $\mathbf{x}^1$ by a rotation around the origin by $2(i-1)\pi/n$, it suffices to analyze $\mathbf{x}^1$ for regularity and injectivity. This is conveniently done using the Bézier representation of $\mathbf{x}^1$.

We demonstrate this for the case $k = 2$ exemplarily. The Bézier points of $\mathbf{x}^1$ are shown in Figure 4.2.

$$
\begin{array}{c}
\begin{bmatrix} c_n^1 + \frac{1}{6}c_n^3 \\ \frac{2}{3}s_n^1 + \frac{1}{6}s_n^3 \end{bmatrix} \\
\begin{bmatrix} c_n^1 + \frac{1}{24}c_n^3 \\ \frac{5}{12}s_n^1 + \frac{1}{24}s_n^3 \end{bmatrix} \\
\begin{bmatrix} c_n^1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} c_n^1 + \frac{1}{24}c_n^3 \\ -\frac{5}{12}s_n^1 - \frac{1}{24}s_n^3 \end{bmatrix} \\
\begin{bmatrix} c_n^1 + \frac{1}{6}c_n^3 \\ -\frac{2}{3}s_n^1 - \frac{1}{6}s_n^3 \end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{bmatrix} \frac{5}{4}c_n^1 + \frac{1}{6}c_n^3 \\ \frac{11}{12}s_n^1 + \frac{1}{6}s_n^3 \end{bmatrix} \\
\begin{bmatrix} \frac{5}{4}c_n^1 + \frac{1}{24}c_n^3 \\ \frac{2}{3}s_n^1 + \frac{1}{24}s_n^3 \end{bmatrix} \\
\begin{bmatrix} \frac{5}{4}c_n^1 \\ \frac{1}{4}s_n^1 \end{bmatrix} \\
\begin{bmatrix} \frac{5}{4}c_n^1 \\ -\frac{1}{4}s_n^1 \end{bmatrix} \\
\begin{bmatrix} \frac{5}{4}c_n^1 + \frac{1}{24}c_n^3 \\ -\frac{2}{3}s_n^1 - \frac{1}{24}s_n^3 \end{bmatrix} \\
\begin{bmatrix} \frac{5}{4}c_n^1 + \frac{1}{6}c_n^3 \\ -\frac{11}{12}s_n^1 - \frac{1}{6}s_n^3 \end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{bmatrix} \frac{3}{2}c_n^1 + \frac{1}{6}c_n^3 \\ \frac{7}{6}s_n^1 + \frac{1}{6}s_n^3 \end{bmatrix} \\
\begin{bmatrix} \frac{3}{2}c_n^1 + \frac{1}{24}c_n^3 \\ \frac{11}{12}s_n^1 + \frac{1}{24}s_n^3 \end{bmatrix} \\
\begin{bmatrix} \frac{3}{2}c_n^1 \\ \frac{1}{2}s_n^1 \end{bmatrix} \\
\begin{bmatrix} \frac{3}{2}c_n^1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} \frac{3}{2}c_n^1 \\ -\frac{1}{2}s_n^1 \end{bmatrix} \\
\begin{bmatrix} \frac{3}{2}c_n^1 + \frac{1}{24}c_n^3 \\ -\frac{11}{12}s_n^1 - \frac{1}{24}s_n^3 \end{bmatrix} \\
\begin{bmatrix} \frac{3}{2}c_n^1 + \frac{1}{6}c_n^3 \\ -\frac{7}{6}s_n^1 - \frac{1}{6}s_n^3 \end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{bmatrix} \frac{7}{4}c_n^1 + \frac{1}{6}c_n^3 \\ \frac{17}{12}s_n^1 + \frac{1}{6}s_n^3 \end{bmatrix} \\
\begin{bmatrix} \frac{7}{4}c_n^1 + \frac{1}{24}c_n^3 \\ \frac{7}{6}s_n^1 + \frac{1}{24}s_n^3 \end{bmatrix} \\
\begin{bmatrix} \frac{7}{4}c_n^1 \\ \frac{3}{4}s_n^1 \end{bmatrix} \\
\begin{bmatrix} \frac{7}{4}c_n^1 \\ \frac{1}{4}s_n^1 \end{bmatrix} \\
\begin{bmatrix} \frac{7}{4}c_n^1 \\ -\frac{1}{4}s_n^1 \end{bmatrix} \\
\begin{bmatrix} \frac{7}{4}c_n^1 \\ -\frac{1}{4}s_n^1 \end{bmatrix} \\
\begin{bmatrix} \frac{7}{4}c_n^1 + \frac{1}{24}c_n^3 \\ -\frac{7}{6}s_n^1 - \frac{1}{24}s_n^3 \end{bmatrix} \\
\begin{bmatrix} \frac{7}{4}c_n^1 + \frac{1}{6}c_n^3 \\ -\frac{17}{12}s_n^1 - \frac{1}{6}s_n^3 \end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{bmatrix} 2c_n^1 + \frac{1}{6}c_n^3 \\ \frac{5}{3}s_n^1 + \frac{1}{6}s_n^3 \end{bmatrix} \\
\begin{bmatrix} 2c_n^1 + \frac{1}{24}c_n^3 \\ \frac{17}{12}s_n^1 + \frac{1}{24}s_n^3 \end{bmatrix} \\
\begin{bmatrix} 2c_n^1 \\ s_n^1 \end{bmatrix} \\
\begin{bmatrix} 2c_n^1 \\ \frac{1}{2}s_n^1 \end{bmatrix} \\
\begin{bmatrix} 2c_n^1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2c_n^1 \\ -\frac{1}{2}s_n^1 \end{bmatrix} \\
\begin{bmatrix} 2c_n^1 \\ -s_n^1 \end{bmatrix} \\
\begin{bmatrix} 2c_n^1 + \frac{1}{24}c_n^3 \\ -\frac{17}{12}s_n^1 - \frac{1}{24}s_n^3 \end{bmatrix} \\
\begin{bmatrix} 2c_n^1 + \frac{1}{6}c_n^3 \\ -\frac{5}{3}s_n^1 - \frac{1}{6}s_n^3 \end{bmatrix}
\end{array}
$$

FIG. 4.2. *The Bézier points of* $\mathbf{x}^1$ *for* $k = 2$ *with* $c_n^i := \cos(i\pi/n), s_n^i := \sin(i\pi/n), i = 1, 3$.

The differences of neighbouring Bézier points converge to the directional derivatives with respect to $\pm\mathbf{v}_1, \pm\mathbf{v}_2$ or $\pm\mathbf{v}_3$, if the Bézier net is refined by the de-Casteljau-algorithm. Due to the form of $\mathcal{X}^1$ is follows that none of these directional derivatives vanishes. Furthermore, a computation with Mathematica gives the following intervals B_1, B_2 and B_3 for the angles between the 1-axis and the directional derivatives with respect to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$, respectively:

$$
B_1 \subset \begin{cases} [-60^\circ, 0^\circ] & \text{for } n = 3 \\ [-45^\circ, 0^\circ) & \text{for } n \geq 4 \end{cases},
$$
$$
B_2 \subset \begin{cases} [60^\circ, 120^\circ] & \text{for } n = 3 \\ (0^\circ, 180^\circ) & \text{for } n \geq 4 \end{cases},
$$
$$
B_3 \subset \begin{cases} [0^\circ, 60^\circ] & \text{for } n = 3 \\ (0^\circ, 45^\circ] & \text{for } n \geq 4 \end{cases}.
$$

If $\mathbf{x}^1$ were not regular, all three directional derivatives would be linearly dependent. In this case they would have either the same or the opposite orientation, i.e. the angle between them and the 1-axis would be either the same or enlarged by 180°. Note that the set

$$
\begin{array}{lcl}
(B_1 \cap B_2 \cap B_3) & \cup & (B_1 \cap R(B_2) \cap B_3) \;\cup \\
(B_1 \cap B_2 \cap R(B_3)) & \cup & (B_1 \cap R(B_2) \cap R(B_3))
\end{array}
\qquad (4.1)
$$

is empty for $n \geq 4$, where $R(B)$ denotes the interval that results form the interval B by adding $180°$ to both interval bounds. So always two directional derivatives are linearly independent. Therefore $\mathbf{x}^1$ is globally regular for $n \geq 4$.

For $n = 3$ the set (4.1) is not empty. One refinement of the Bézier net with the de-Casteljau-algorithm results in a finer Bézier net for $\mathbf{x}^1$. The Bézier points of the finer net are computed by convex combinations of the old Bézier points. Thus the intervals for the angles between the directional derivatives and the 1-axis shrink, i.e. the new intervals are subsets of the interior of the old intervals. Obviously for these new intervals the set (4.1) is empty.

This proves that $\mathbf{x}^1$ is regular for $n \geq 3$.

To prove injectivity of $\mathbf{x}^1$ let $\mathbf{a}$ and $\mathbf{b}$ be to different points in the parameter domain of $\mathbf{x}^1$ and assume that $\mathbf{x}^1(\mathbf{a}) = \mathbf{x}^1(\mathbf{b})$. Then $\mathbf{x}^1(t\mathbf{a} + (1-t)\mathbf{b}), t \in [0,1]$, is a closed curve on $\mathbf{x}^1$, whose tangent must cover an angle of at least $180°$. This tangent can be represented as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$. Without loss of generality set $\mathbf{a} - \mathbf{b} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_3$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \geq 0$. For $\alpha \geq 0$ the angles between the tangent and the 1-axis lie in $B_1 \cup B_3 \subset [-60°, 60°]$, which is shorter than $180°$. An analogous result holds for $\alpha < 0$. Therefore the assumption must be wrong that such two points $\mathbf{a}$ and $\mathbf{b}$ exist and the segment $\mathbf{x}^1$ is injective.

Thus the surface ring defined by $\mathcal{C}$ is a regular and injective C^k-surface.

For the construction of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ note, that the three outer rings of Bézier points of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are determined by the C^0-, C^1- and C^2-transition to $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$, see Figure 4.3. The common Bézier point of all n patches $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is for symmetry reasons the origin. Thus there remains only one Bézier points, which can be chosen for example as $[b_1\, b_2]^t = [\frac{1}{4}c_n^1 + \frac{1}{6}c_n^3, \frac{1}{12}(s_n^1 + s_n^3)]^t$, see Figure 4.3. The same argument as above proves that $\mathbf{x}_1$ is regular and injective. Furthermore, this choice guarantees that $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_n$ do not overlap.

$$
\begin{array}{ccccc}
 & & & & \begin{bmatrix} c_n^1 + \frac{1}{6}c_n^3 \\ \frac{2}{3}s_n^1 + \frac{1}{6}s_n^3 \end{bmatrix} \\
 & & & \begin{bmatrix} \frac{3}{4}c_n^1 + \frac{1}{6}c_n^3 \\ \frac{5}{12}s_n^1 + \frac{1}{6}s_n^3 \end{bmatrix} & \begin{bmatrix} c_n^1 + \frac{1}{24}c_n^3 \\ \frac{5}{12}s_n^1 + \frac{1}{24}s_n^3 \end{bmatrix} \\
 & & \begin{bmatrix} \frac{1}{2}c_n^1 + \frac{1}{6}c_n^3 \\ \frac{1}{6}s_n^1 + \frac{1}{6}s_n^3 \end{bmatrix} & \begin{bmatrix} \frac{3}{4}c_n^1 + \frac{1}{24}c_n^3 \\ \frac{1}{6}s_n^1 + \frac{1}{24}s_n^3 \end{bmatrix} & \\
 & \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} & & & \begin{bmatrix} c_n^1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 \\ 0 \end{bmatrix} & & \begin{bmatrix} \frac{1}{2}c_n^1 + \frac{1}{12}c_n^3 \\ 0 \end{bmatrix} & & \\
 & \begin{bmatrix} b_1 \\ -b_2 \end{bmatrix} & & \begin{bmatrix} \frac{3}{4}c_n^1 + \frac{1}{24}c_n^3 \\ -\frac{1}{6}s_n^1 - \frac{1}{24}s_n^3 \end{bmatrix} & \begin{bmatrix} c_n^1 + \frac{1}{24}c_n^3 \\ -\frac{5}{12}s_n^1 - \frac{1}{24}s_n^3 \end{bmatrix} \\
 & & \begin{bmatrix} \frac{1}{2}c_n^1 + \frac{1}{6}c_n^3 \\ -\frac{1}{6}s_n^1 - \frac{1}{6}s_n^3 \end{bmatrix} & \begin{bmatrix} \frac{3}{4}c_n^1 + \frac{1}{6}c_n^3 \\ -\frac{5}{12}s_n^1 - \frac{1}{6}s_n^3 \end{bmatrix} & \begin{bmatrix} c_n^1 + \frac{1}{6}c_n^3 \\ -\frac{2}{3}s_n^1 - \frac{1}{6}s_n^3 \end{bmatrix}
\end{array}
$$

FIG. 4.3. *The Bézier points of* $\mathbf{x}_1$ *for* $k = 2$ *with* $c_n^i := \cos(i\pi/n), s_n^i := \sin(i\pi/n), i = 1, 3$.

Summing up, we have proved the following Lemma:

LEMMA 4.1. *The control net* $\mathcal{C}$ *together with the Bézier point* $[b_1\, b_2]^t$ *define a regular and injective plane parametrization* $\mathbf{x}$. *The resulting p-patch is a regular* G^2-*surface.*

REMARK 4.2. *The prove for the existence of a regular, injective plane parametrization with* $k/2 - 1$ *coalescing rings of control points for arbitrary even* $k \geq 4$ *can be performed in a similar fashion. The prove for* $k = 4$ *can be found in [22].*

For the construction of a plane parametrization in the half-box spline case we take the dual net of the control net in Figure 4.1. The resulting hexagonal net is the union of two triangular nets: one control net for the half-box splines $H_m^{\triangle}$ and one for the half-box splines H_m^{∇}. These two nets determine two half-box spline surfaces whose sum is the half-box spline surface determined by the hexagonal net.

The two triangular nets are the nets connecting midpoints of triangles of the primal net that share a vertex but not an edge. Therefore the angles between the directions of their edges and the 1-axis are the same as for the primal triangular net. Because each of the two half-box spline surfaces can be transformed to Bézier form separately ([1]) and the Bézier net of the complete half-box spline surface is the average of the Bézier nets of the separate half-box spline surfaces, the angles between the directional derivatives and the 1-axis must lie in the same intervals as for the corresponding box spline surface of the primal triangular net.

For $k = 1$ this yields that the half-box spline surface $\mathbf{x}_{n+1} \cup \cdots \cup \mathbf{x}_{4n}$ of the dual of the net $\mathcal{C}$ is regular and injective for $n \geq 3$. For the construction of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ note that the two outer rings of Bézier points of $\mathbf{x}_1$ are convex combinations of control points with positive first coordinate of the dual of $\mathcal{X}^1$. Thus these Bézier points must all lie right of the origin, so that the remaining Bézier point $[b_1\, b_2]^t$ can be chosen appropriately.

LEMMA 4.3. *For the dual of the net $\mathcal{C}$ the remaining Bézier point $[b_1\, b_2]^t$ can be chosen such that a regular and injective plane parametrization $\mathbf{x}$ is defined. The resulting p-patch is a regular G^1-surface.*

REMARK 4.4. *In the same way as above the proof for G^4 carries over to G^3.*

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