Quadric Splines

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1 Introduction

Surface rendering or point location on a surface can easier be accomplished in an implicit rather than parametric representation. This observation has been the key motivation for developing piecewise algebraic splines.
In particular, Dahmen [1989] and Guo [1991] used triangular segments of quadrics to build tangent plane continuous surfaces interpolating the vertices of a triangular net with prescribed normals. Their construction is based on the implicit Bézier representation introduced by Sederberg [1985] and employs the idea of the Powell-Sabin split [1977] for bivariate C^1-piecewise quadratics.
While Dahmen's and Guo's approach is completely algebraic, the objective of this paper is to derive their quadric splines solely geometrically in projective space. The geometric approach has several benefits. It provides a geometric meaning for certain parameters chosen to be the same constant by Dahmen and Guo. Furthermore, it facilitates the classification of the quadrics, avoids the global dependencies of Dahmen's and Guo's transversal system, and renders the Powell-Sabin interpolant as a special case.

2 Preliminaries

Throughout the paper small hollow letters, $\mathbb{a}$, $\mathbb{b}$,..., are used to denote points in projective 3-space while capital script, $\mathcal{A}$, $\mathcal{B}$,..., is used to denote (the equations of) planes and quadrics. A point of a quadric and the tangent plane at this point are always denoted by the same letter, i.e. by $\mathbb{p}$ and $\mathcal{P}$ (or $\mathcal{P}(\mathbb{x}) = 0$). Together they form a **contact element** which is briefly referred to as the contact element $\boxed{\mathbb{p}}$.

Before we construct a quadric spline we observe that there is in general no single quadric with three arbitrary contact elements $\boxed{\mathbb{a}}$, $\boxed{\mathbb{b}}$, $\boxed{\mathbb{c}}$. Such a quadric exists only if there is a conic through $\boxed{\mathbb{a}}$, $\boxed{\mathbb{b}}$, $\boxed{\mathbb{c}}$ (in the plane $\mathbb{abc}$).

However, three arbitrary contact elements can always be interpolated by a C^1-**macro patch** consisting of 6 triangular quadric segments as illustrated in Figure 1. Fewer segments do not suffice. This is obvious for three segments arranged as in Figure 1, middle (since the three conics through any two vertices and the vertex $\mathbb{p}$ common to all segments would lie on a single quadric) and can also be shown for four segments as in Figure 1, right.

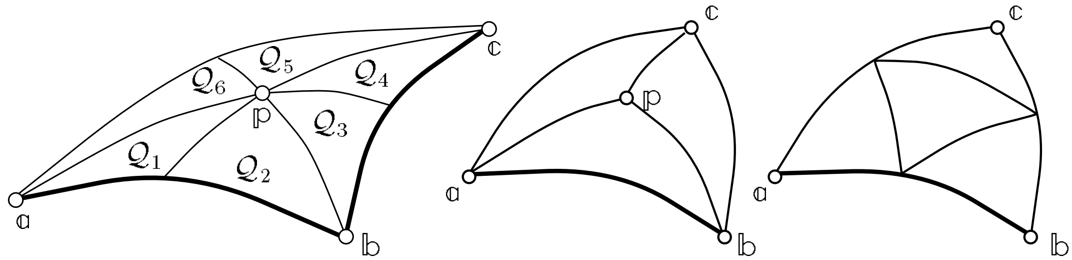

Figure 1: A macro patch (left) and two stiff configurations (right).

Remark 2.1 *All quadrics with three common contact elements* $\boxed{\mathbb{a}}$, $\boxed{\mathbb{b}}$, $\boxed{\mathbb{c}}$ *are members of the pencil* $\mathcal{K} + \lambda \mathcal{U}^2$ *where* $\mathcal{U}$ *denotes the plane* $\mathbb{a}\mathbb{b}\mathbb{c}$ *and* $\mathcal{K}$ *the unique tangent cone through* $\boxed{\mathbb{a}}$, $\boxed{\mathbb{b}}$, *and* $\boxed{\mathbb{c}}$. *In particular, two quadrics with* C^1*-contact along a curve are members of such a pencil and vice versa. Hence the common curve is a (double) conic.*

Remark 2.2 *A conic through* $\boxed{\mathbb{a}}$, $\boxed{\mathbb{b}}$, *and* $\boxed{\mathbb{c}}$ *exists if and only if* $\boxed{\mathbb{a}}$, $\boxed{\mathbb{b}}$, *and* $\boxed{\mathbb{c}}$ *form Brianchon's configuration as illustrated in Figure 2. This condition can be expressed algebraically as*

$$\mathcal{A}(\mathbb{b})\mathcal{B}(\mathbb{c})\mathcal{C}(\mathbb{a}) \;=\; \mathcal{B}(\mathbb{a})\mathcal{C}(\mathbb{b})\mathcal{A}(\mathbb{c}).$$

Guo [1991] *derives an equivalent form from the algebraic equation of a quadric through* $\boxed{\mathbb{a}}$, $\boxed{\mathbb{b}}$, *and* $\boxed{\mathbb{c}}$.

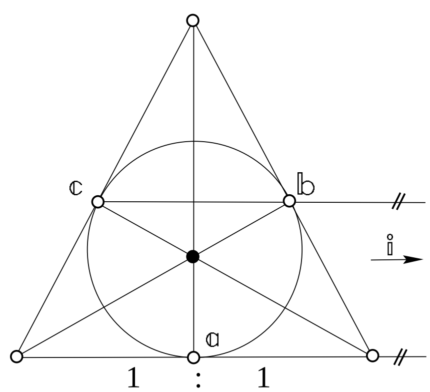

Figure 2: Brianchon's configuration.

Remark 2.3 *Mapping the conic* $\boxed{\mathbb{a}}$ $\boxed{\mathbb{b}}$ $\boxed{\mathbb{c}}$ *onto a circle such that the line* $\mathbb{b}\,\mathbb{c}$ *meets the tangent in* $\mathbb{a}$ *in an ideal point* $\mathbb{i}$ *shows that a conic through three contact elements* $\boxed{\mathbb{a}}$, $\boxed{\mathbb{b}}$, *and* $\boxed{\mathbb{c}}$ *exists if and only if* $\mathbb{a}$ *and* $\mathbb{i}$ *are separated harmonically by the tangents in* $\mathbb{c}$ *and* $\mathbb{b}$, *cf. Figure 2.*

3 Biarcs

A planar cut through two quadric segments with C^1-contact along a conic gives a **biarc** consisting of two conics touching each other in two points $\mathbb{p}$ and $\mathbb{q}$ as illustrated in Figure 3.

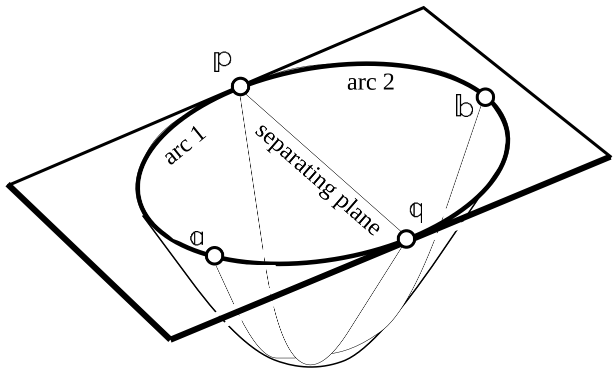

Figure 3: A planar cut.

As a consequence of Remark 2.3 the biarc is completely determined by the **separating line** $\mathbb{p}\mathbb{q}$, a common contact element, say $\boxed{\mathbb{p}}$, a contact element $\boxed{\mathbb{a}}$ on the first and a contact element $\boxed{\mathbb{b}}$ on the second arc. This construction of a biarc is shown in Figure 4, where the given elements are marked by hollow and the constructed elements by solid dots.

4 Constructing the Macro Patch

The simple biarc construction is essentially all that is needed to construct a C^1-macro patch consisting of six quadric segments $\mathcal{Q}_1, \ldots, \mathcal{Q}_6$ with one common contact element $\boxed{\mathbb{p}}$ as shown in Figure 1. Following Dahmen and Guo we choose the six planes separating the six quadric segments so as to meet in some so called **transversal line** $\mathcal{L}$ through $\boxed{\mathbb{p}}$.

The six quadrics to be constructed have a second contact element $\boxed{\mathbb{q}}$ on $\mathcal{L}$. Let $\bar{\mathbb{a}}$, $\bar{\mathbb{b}}$, and $\bar{\mathbb{c}}$ be the intersections of $\mathcal{L}$ with $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ as illustrated in Figure 5. It follows from Remark 2.3 that the tangent planes $\mathcal{P}$ and $\mathcal{Q}$ divide the pairs $\mathbb{a}\bar{\mathbb{a}}$, $\mathbb{b}\bar{\mathbb{b}}$, and $\mathbb{c}\bar{\mathbb{c}}$ harmonically. Hence we may choose $\boxed{\mathbb{p}}$ arbitrarily and construct $\boxed{\mathbb{q}}$ or vice versa.

Lemma 4.1 *Any three contact elements* $\boxed{\mathbb{a}}$, $\boxed{\mathbb{p}}$, $\boxed{\mathbb{b}}$, *and any point* $\mathbb{x}$ *determine two unique quadrics* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *through* $\boxed{\mathbb{a}}$, $\boxed{\mathbb{p}}$, $\mathbb{x}$, *and* $\boxed{\mathbb{b}}$, $\boxed{\mathbb{p}}$, $\mathbb{x}$, *respectively, with* C^1*-contact in any prescribed plane* $\mathcal{U}$ *through* $\mathbb{p}$ *and* $\mathbb{x}$, *see Figure 3 for an illustration.*

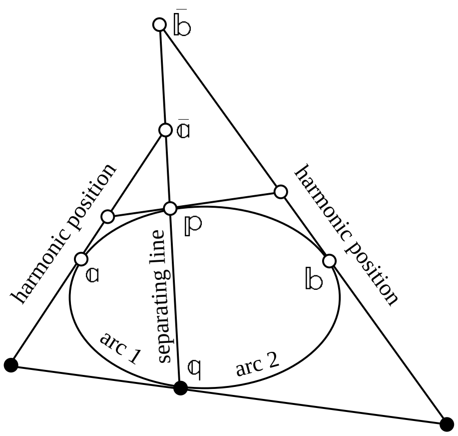

Figure 4: Biarc construction.

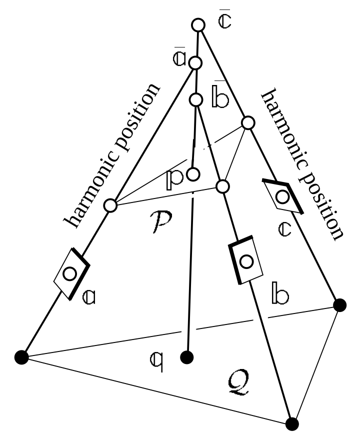

Figure 5: Constructing the second common contact element.

Proof:
The two conics $\boxed{\mathbb{a}}\,\boxed{\mathbb{p}}\,\mathbb{x}$ and $\boxed{\mathbb{b}}\,\boxed{\mathbb{p}}\,\mathbb{x}$ define the tangent plane $\mathcal{Q}$ at $\mathbb{x}$. The biarc $\boxed{\mathbb{a}}\,\boxed{\mathbb{p}}\,\boxed{\mathbb{b}}$ meets its separating plane $\mathcal{U}$ in $\mathbb{p}$ and a second point $\mathbb{q}$. In both pencils $\boxed{\mathbb{a}}\,\boxed{\mathbb{p}}\,\boxed{\mathbb{x}}$ and $\boxed{\mathbb{b}}\,\boxed{\mathbb{p}}\,\boxed{\mathbb{x}}$ there is each a unique quadric through $\mathbb{q}$. These quadrics carry the biarc and the conic $\boxed{\mathbb{p}}\,\boxed{\mathbb{x}}\,\mathbb{q}$. Thus they have three common contact elements, namely $\boxed{\mathbb{p}}$, $\boxed{\mathbb{x}}$, and $\boxed{\mathbb{q}}$ which implies C^1-contact in $\mathcal{U}$. □

Constructing the four remaining quadrics $\mathcal{Q}_3,\ldots,\mathcal{Q}_6$ analogously as in Lemma 4.1 results then in a C^1-macro patch. Namely $\mathcal{Q}_1$ and $\mathcal{Q}_6$, for example, have three common contact elements $\boxed{\mathbb{a}}$, $\boxed{\mathbb{p}}$, $\boxed{\mathbb{q}}$ and hence C^1-contact in the plane $\mathbb{a}\mathbb{p}\mathbb{q}$, see Remark 2.1.

In the following we will always assume that the three boundary curves $\mathbb{a}\mathbb{b}$, $\mathbb{b}\mathbb{c}$, $\mathbb{c}\mathbb{a}$ of a macro patch are planar biarcs.

Remark 4.2 *In case $\mathbb{p}$ is the ideal point of the z-axis and $\mathcal{P}$ the ideal plane, all 6 quadrics $\mathcal{Q}_1,\ldots,\mathcal{Q}_6$ are paraboloids defining quadratic polynomials over the xy-plane. Thus, if the three boundary planes of the macro patch are chosen to be parallel to the z-axis, the macro patch is a Powell-Sabin element* [Powell & Sabin 1977]*. See also Remarks 7.2 and 7.3.*

5 C^1-Propagation

Two macro patches with vertices $\mathbb{a}$, $\mathbb{b}$, $\mathbb{c}$, and $\bar{\mathbb{c}}$ as illustrated in Figure 6 have in general two different boundary curves $\mathbb{a}\mathbb{b}$, even if these curves lie in the same plane. The following two properties are employed in the next section to fill this gap smoothly:

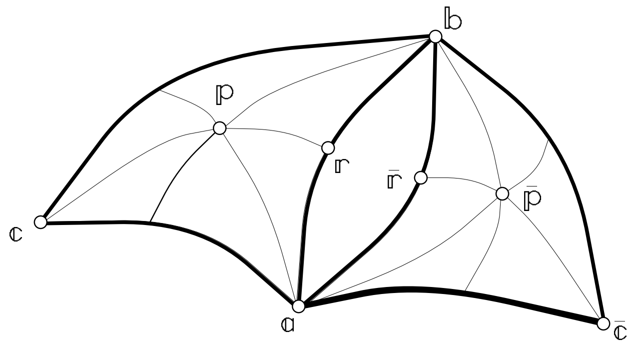

Figure 6: Two adjacent macro patches.

Two quadrics have C^1-contact in a plane $\mathcal{U}$ if and only if they lie in the pencil $\mathcal{Q} + \lambda\mathcal{U}^2$, $\lambda \in \mathbb{R}_\infty$, where $\mathcal{Q}$ denotes one of these two quadrics. This C^1-contact in $\mathcal{U}$ is inherited by all pairs of quadrics

$$\mathcal{Q} + \mu\mathcal{V}^2 \quad \text{and} \quad (\mathcal{Q} + \lambda\mathcal{U}^2) + \mu\mathcal{V}^2$$

having C^1-contact with $\mathcal{Q}$ and $\mathcal{Q} + \lambda\mathcal{U}^2$ in any plane $\mathcal{V}$ since $(\mathcal{Q} + \lambda\mathcal{U}^2) + \mu\mathcal{V}^2 = (\mathcal{Q} + \mu\mathcal{V}^2) + \lambda\mathcal{U}^2$. Note that for every point $\mathbb{x}$ in $\mathcal{U}$ there is exactly one such pair of quadrics containing $\mathbb{x}$ (see Figure 7).

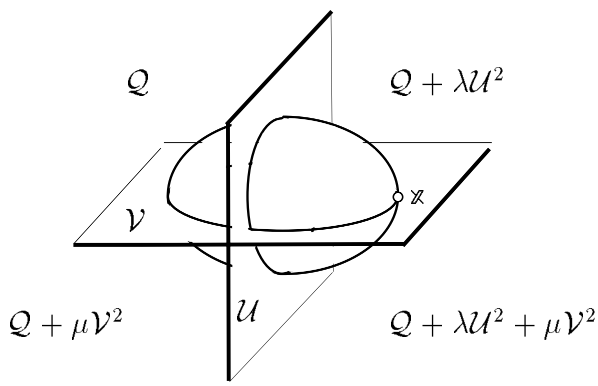

Figure 7: C^1-propagation

6 Connecting the Macro Patches

A smooth connection between two macro patches can be constructed using four intermediate quadrics $\mathcal{R}_1$, $\mathcal{R}_2$, $\bar{\mathcal{R}}_1$, $\bar{\mathcal{R}}_2$ as illustrated in Figure 8.

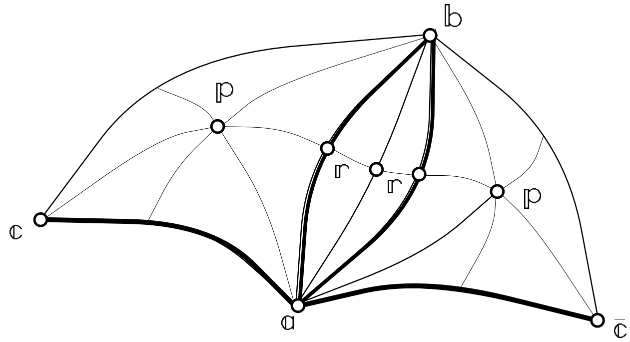

Figure 8: Filling the gap between two adjacent macro patches

Following Dahmen and Guo we assume that the quadric segments $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\bar{\mathcal{Q}}_1$, $\bar{\mathcal{Q}}_2$ of the macro patches are separated by a common plane $\mathcal{U}$ which implies the coplanarity of the transversal lines $\mathcal{L}$ and $\bar{\mathcal{L}}$. Let $\mathbb{r}$ and $\bar{\mathbb{r}}$ be the two points lying in $\mathcal{U}$ and on the boundary curves $\mathbb{a}\mathbb{b}$ of the two macro patches, respectively. Then the gap can be filled by the following construction:

Construction 6.1

Choose any plane $\mathcal{V}$ through $\mathbb{a}\mathbb{b}$ separating both macro patches. The biarc $\boxed{\mathbb{r}}\boxed{\mathbb{a}}\boxed{\bar{\mathbb{r}}}$ with separating plane $\mathcal{V}$ intersects $\mathcal{V}$ in $\mathbb{a}$ and a second point $\mathbb{s}$.

Let $\mathcal{R}_1$ be the quadric through $\mathbb{s}$ having C^1-contact with $\mathcal{Q}_1$ in the plane $\mathbb{a}\mathbb{r}\mathbb{b}$ and let $\bar{\mathcal{R}}_1$ be the quadric through $\bar{\mathbb{r}}$ which has C^1-contact with $\mathcal{R}_1$ in the separating plane $\mathcal{V}$.

As explained in Section 5 there exist unique quadrics $\mathcal{R}_2$ and $\bar{\mathcal{R}}_2$ through $\mathbb{b}$ having C^1-contact with $\mathcal{R}_1$ and $\bar{\mathcal{R}}_1$ in $\mathcal{U}$, where $\mathcal{R}_2$ has also C^1-contact with $\mathcal{Q}_2$ and $\bar{\mathcal{R}}_2$ in the planes $\mathbb{a}\mathbb{r}\mathbb{b}$ and $\mathcal{V}$, respectively.

7 The Smoothness of the Filling

The quadrics $\mathcal{R}_1$, $\mathcal{R}_2$, $\bar{\mathcal{R}}_1$, $\bar{\mathcal{R}}_2$ establish a smooth (i.e. tangent plane continuous) filling between the macro patches if also $\bar{\mathcal{R}}_1$ and $\bar{\mathcal{Q}}_1$ have C^1-contact. The theorem below gives a simple criterion for this C^1-contact.

Theorem 7.1 *For any contact element $\boxed{\mathbb{q}}$ let $\mathbb{x}_q$ and $\mathbb{y}_q$ denote the intersections of the line $\mathbb{a}\mathbb{b}$ with the tangent plane $\mathcal{Q}$ and the plane of the pencil $\mathcal{AB}$ through $\mathbb{q}$, respectively. Then the quadrics $\bar{\mathcal{R}}_1$ and $\bar{\mathcal{Q}}_1$ have C^1-contact if and only if the cross ratios*

$$cr\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_p\,\mathbb{y}_p] \;=\; cr\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_{\bar{p}}\,\mathbb{y}_{\bar{p}}]$$

are identical.

Remark 7.2 *If* $\boxed{\mathbb{p}} = \boxed{\bar{\mathbb{p}}}$*, no extra filling is needed. Namely,* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *are uniquely determined by* $\boxed{\mathbb{a}}$*,* $\boxed{\mathbb{b}}$*,* $\boxed{\bar{\mathbb{p}}}$*, a further point* $\mathbb{q}$ *in* $\mathcal{U}$*, and their* C^1*-contact in* $\mathcal{U}$*, see Lemma 4.1. Hence* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *are also the unique quadrics through* $\mathbb{q}$ *having* C^1*-contact with* $\bar{\mathcal{Q}}_1$ *and* $\bar{\mathcal{Q}}_2$ *in the plane* $\mathbb{a}\mathbb{b}\bar{\mathbb{p}}$ *as explained in Section 5.*

Remark 7.3 *In particular, if all macro patches are Powell Sabin elements, see Remark 4.2, no filling is needed.*

8 Proof of the Theorem

For the proof of Theorem 7.1 we revisite Lemma 4.1 and use the same notation as in Figure 5. Let π be the unique projective map with fixed points $\mathbb{a}$, $\bar{\mathbb{a}}$, $\mathbb{b}$, and $\bar{\mathbb{b}}$ which maps any plane $\mathcal{P}$' through $\mathbb{p}$ onto the plane $\mathcal{Q}$' through $\mathbb{q}$ such that $\mathcal{P}$' and $\mathcal{Q}$' separate both pairs $\mathbb{a}\bar{\mathbb{a}}$ and $\mathbb{b}\bar{\mathbb{b}}$ harmonically. Thus Lemma 4.1 can be generalized as follows:

Lemma 8.1 *There are two quadrics through* $\boxed{\mathbb{a}}$*,* $\boxed{\mathbb{p}}$*,* $\boxed{\mathbb{q}}$ *and* $\boxed{\mathbb{b}}$*,* $\boxed{\mathbb{p}}$*,* $\boxed{\mathbb{q}}$*, respectively, with* C^1*-contact in any prescribed plane* $\mathcal{U}$ *through* $\mathbb{p}$ *and* $\mathbb{q}$ *if and only if* $\pi(\mathcal{P}) = \mathcal{Q}$*. These quadrics are unique.*

Furthermore, the tangents of the conics $\boxed{\mathbb{a}}\ \mathbb{p}\ \boxed{\mathbb{b}}$ and $\boxed{\mathbb{a}}\ \mathbb{q}\ \boxed{\mathbb{b}}$ at $\mathbb{p}$ and $\mathbb{q}$, respectively, intersect the line $\mathbb{a}\mathbb{b}$ in two points $\mathbb{z}_p$ and $\mathbb{z}_q$, respectively. If $\mathcal{P}$ contains $\mathbb{z}_p$, then there exists a single quadric through $\boxed{\mathbb{a}}$, $\boxed{\mathbb{p}}$, $\boxed{\mathbb{b}}$, and $\mathbb{q}$. Hence $\mathcal{Q}$ contains $\mathbb{z}_q$. Figure 2 shows that the points $\mathbb{a}$, $\mathbb{b}$, $\mathbb{y}_p$, and $\mathbb{z}_p$ are in harmonic position, and similarly $\mathbb{a}$, $\mathbb{b}$, $\mathbb{y}_q$, and $\mathbb{z}_q$.

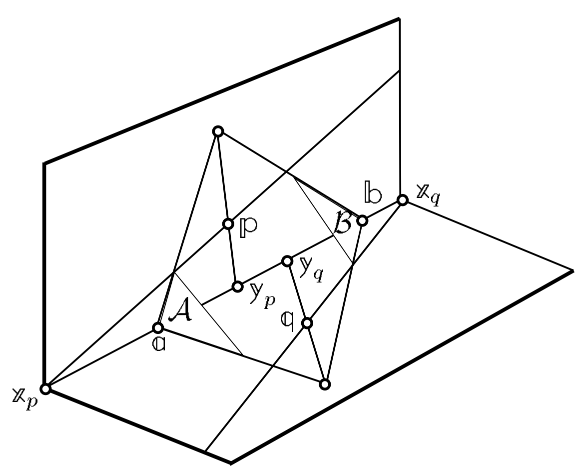

Figure 9: Cross Ratios

Thus, π maps $\mathbb{a}$, $\mathbb{b}$, $\mathbb{x}_p$, $\mathbb{y}_p$ onto $\mathbb{a}$, $\mathbb{b}$, $\mathbb{x}_q$, $\mathbb{y}_q$, respectively, which implies

$$\mathrm{cr}\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_p\,\mathbb{y}_p] \;=\; \mathrm{cr}\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_q\,\mathbb{y}_q].$$

So, if $\mathcal{R}_1$, $\bar{\mathcal{R}}_1$, $\bar{\mathcal{R}}_2$, $\mathcal{R}_2$ define a smooth filling, Lemma 8.1 implies

$$\mathrm{cr}\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_p\,\mathbb{y}_p] \;=\; \mathrm{cr}\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_r\,\mathbb{y}_r] \;=\; \mathrm{cr}\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_{\bar r}\,\mathbb{y}_{\bar r}] \;=\; \mathrm{cr}\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_{\bar p}\,\mathbb{y}_{\bar p}].$$

On the other hand, if the cross ratios are equal, the tangent planes of $\bar{\mathcal{R}}_1$ and $\bar{\mathcal{Q}}_1$ at $\bar{\mathbb{r}}$ contain both the same point $\mathbb{x}_{\bar r}$ and contact the biarc $\boxed{\mathbb{r}}\boxed{\mathbb{a}}\boxed{\bar{\mathbb{r}}}$. Hence they are identical.
Now substituting $\bar{\mathcal{R}}$ and $\bar{\mathbb{r}}$ for $\mathcal{P}$ and $\mathbb{p}$ in Remark 7.2 gives for $i = 1, 2$ that $\bar{\mathcal{R}}_i$ has C^1-contact with $\bar{\mathcal{Q}}_i$ in the plane $\mathbb{a}\bar{\mathbb{r}}\mathbb{b}$.

9 Some Remarks

Remark 9.1 *The map π can be expressed in homogeneous coordinates as:*

$$\begin{aligned}\mathcal{P} \;\mapsto\; \mathcal{Q} \;=\; & \;\mathcal{A}(\mathbb{q})\,\mathcal{B}(\mathbb{p})\,\mathcal{P}(\mathbb{a}) \;\cdot\; (\mathbb{b}\wedge\mathbb{p}\wedge\mathbb{q})\\ & +\mathcal{A}(\mathbb{p})\,\mathcal{B}(\mathbb{q})\,\mathcal{P}(\mathbb{b}) \;\cdot\; (\mathbb{a}\wedge\mathbb{p}\wedge\mathbb{q})\\ & +\mathcal{A}(\mathbb{p})\,\mathcal{B}(\mathbb{p})\,\mathcal{P}(\mathbb{q}) \;\cdot\; (\mathbb{a}\wedge\mathbb{b}\wedge\mathbb{q}),\end{aligned}$$

where $\wedge$ denotes the alternating product. This follows from Remark 2.2 and the fact $\mathcal{Q}(\mathbb{q}) = 0$.

Remark 9.2 *By straightforward algebraic manipulations one obtains for the cross ratio of Theorem 7.1:*

$$cr\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_p\,\mathbb{y}_p] \;=\; -\frac{\mathcal{B}(\mathbb{a})\,\mathcal{A}(\mathbb{p})\,\mathcal{P}(\mathbb{b})}{\mathcal{A}(\mathbb{b})\,\mathcal{B}(\mathbb{p})\,\mathcal{P}(\mathbb{a})}.$$

In particular, if $\mathcal{P}(\mathbb{x}) = 0$ is in Hessian normal form, $\mathcal{P}(\mathbb{a})$ represents the Euclidean distance between $\mathbb{a}$ and $\mathcal{P}$. For the other planes holds the analogue.

Remark 9.3 *The macro patch has two contact elements $\boxed{\mathbb{p}}$ and $\boxed{\mathbb{q}}$ on the transversal line $\mathcal{L}$, see section 4. In the proof of Theorem 7.1, $\mathbb{p}$ can denote either one of them. So the cross ratios do not alter if $\mathbb{p}$ and $\mathbb{q}$ are interchanged.*

Remark 9.4 *The product of the three cross ratios on the edges of the triangle $\boxed{\mathbb{a}}\,\boxed{\mathbb{b}}\,\boxed{\mathbb{c}}$ does not depend on $\boxed{\mathbb{p}}$. Let $\mathbb{x}_{ab}$ and $\mathbb{y}_{ab}$ be the intersections of the line $\mathbb{a}\mathbb{b}$ with $\mathcal{P}$ and the plane in the pencil $\mathcal{AB}$ through $\mathbb{p}$, respectively, and define $\mathbb{x}_{bc}$, $\mathbb{y}_{bc}$, $\mathbb{x}_{ca}$, and $\mathbb{y}_{ca}$ analogously. Then we get*

$$\gamma \;=\; cr\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_{ab}\,\mathbb{y}_{ab}] \,\cdot\, cr\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_{bc}\,\mathbb{y}_{bc}] \,\cdot\, cr\,[\mathbb{a}\,\mathbb{b}\,\mathbb{x}_{ca}\,\mathbb{y}_{ca}] \;=\; -\frac{\mathcal{A}(\mathbb{b})\,\mathcal{B}(\mathbb{c})\,\mathcal{C}(\mathbb{a})}{\mathcal{B}(\mathbb{a})\,\mathcal{C}(\mathbb{b})\,\mathcal{A}(\mathbb{c})}\,.$$

Consequently, any three points $\mathbb{z}_{ab}$,$\mathbb{z}_{bc}$,$\mathbb{z}_{ca}$ on the edges $\mathbb{a}\mathbb{b}$, $\mathbb{b}\mathbb{c}$, $\mathbb{c}\mathbb{a}$ are collinear if and only if

$$cr\,[\mathbb{a}\,\mathbb{b}\,\mathbb{z}_{ab}\,\mathbb{y}_{ab}] \,\cdot\, cr\,[\mathbb{a}\,\mathbb{b}\,\mathbb{z}_{bc}\,\mathbb{y}_{bc}] \,\cdot\, cr\,[\mathbb{a}\,\mathbb{b}\,\mathbb{z}_{ca}\,\mathbb{y}_{ca}] \;=\; \gamma\,.$$

This generalizes Menelaos' theorem. In particular, $\gamma = -1$ holds if and only if Brianchon's condition holds, see Remark 2.2.

10 Feasible Choice of the Tangent Planes

Consider a set of triangles $\mathbb{a}_i\,\mathbb{a}_j\,\mathbb{a}_k$, $(ijk) = \mathbb{i} \in I \subset \{1, \ldots, n\}^3$, forming a triangulation of given data points $\mathbb{a}_i$, $i = 1, \ldots, n$. For each macro patch $\mathbb{a}_i\,\mathbb{a}_j\,\mathbb{a}_k$, $(ijk) \in I$, let $\mathbb{p}_{\mathbb{i}}$ be the common vertex of its six quadric segments.

Now we will show how one can obtain a feasible set of tangent planes $\mathcal{P}_{\mathbb{i}}$ in the interior vertices $\mathbb{p}_{\mathbb{i}}$ of the macro patches, i.e. tangent planes such that the cross-ratio condition of Theorem 7.1 is satisfied for all edges of the triangulation.

Let α_i be arbitrary weights associated with the vertices $\mathbb{a}_i$ such that each triangle $\mathbb{a}_i\,\mathbb{a}_j\,\mathbb{a}_k$, $\mathbb{i} = (ijk) \in I$, has at least one vertex with non-zero weight. Then the planes $\mathcal{P}_{\mathbb{i}}$ whose equations solve the linear systems

$$\left\{ \begin{array}{lcl} \mathcal{P}_{\mathbb{i}}(\mathbb{a}_l) & = & \alpha_l \mathcal{A}_l(\mathbb{p}_{\mathbb{i}}), \quad l = i, j, k \\ \mathcal{P}_{\mathbb{i}}(\mathbb{p}_{\mathbb{i}}) & = & 0 \end{array} \right.$$

form a feasible set. Namely the cross ratio of Theorem 7.1 associated with any edge $\mathbb{a}_i\,\mathbb{a}_j$ as obtained from Remark 9.2 is given by

$$\mathrm{cr}\ [\mathbb{a}_i\ \mathbb{a}_j\ \mathbb{x}\ \mathbb{y}] \ = \ -\frac{\alpha_j\ \mathcal{A}_j(\mathbb{a}_i)}{\alpha_i\ \mathcal{A}_i(\mathbb{a}_j)}$$

and depends only on $\boxed{\mathbb{a}_i}$ and $\boxed{\mathbb{a}_j}$.

Remark 10.1 *Every feasible set of tangent planes $\mathcal{P}_{\mathbb{i}}$ can be obtained in this way.*

Remark 10.2 *If all weights α_i are 1 and if the equations of all tangent planes are in Hessian normal form one obtains the quadric splines of Dahmen* [1989].

11 Special transversal systems

For each triangle $\mathbb{a}_i\,\mathbb{a}_j\,\mathbb{a}_k$, $\mathbb{i} = (ijk) \in I$, a transversal line $\mathcal{L}_{\mathbb{i}}$ is needed such that the pairs of transversal lines associated with adjacent triangles are coplanar, see Section 6. Dahmen [1989] lists a few systems of such lines. Another transversal system is formed by the lines

$$\mathcal{L}_{ijk} \ = \ \{\mathbb{x} | \alpha_i \mathcal{A}_i(\mathbb{x}) \ = \ \alpha_j \mathcal{A}_j(\mathbb{x}) \ = \ \alpha_k \mathcal{A}_k(\mathbb{x})\},$$

where the α's are arbitrary constants. Note that $\mathcal{L}_{ijk}$ contains the intersection $\mathcal{A}_i \sqcap \mathcal{A}_j \sqcap \mathcal{A}_k$.

Hence a feasible set of tangent planes $\mathcal{P}_{\mathbb{i}}$ has the property that any pair $\mathcal{P}_{ijk}\ \mathcal{P}_{ijl}$ associated with two adjacent triangles intersects the line $\mathbb{a}_i\,\mathbb{a}_j$ in the same point. Thus a special feasible set is obtained if all $\mathcal{P}_{ijk}$ are parallel to their corresponding triangles $\mathbb{a}_i\,\mathbb{a}_j\,\mathbb{a}_k$.

Remark 11.1 *Moreover if all weighted plane equations $\alpha_i \mathcal{A}_i(\mathbb{x}) = 0$ are in Hessian normal form, $\mathcal{L}_{ijk}$ consists of all points with equal distance to $\mathcal{A}_i$, $\mathcal{A}_j$, and $\mathcal{A}_k$.*

12 Avoiding Global Transversals

The choice of an adequate transversal system is far from being trivial. Even with the local constructions of Section 11 it is not always possible to guarantee that the transversals intersect the corresponding triangles in their interior and that the planes spanned by transversals of adjacent triangles meet the common edge. One way to overcome this difficulty is to replace a given triangle $\mathbb{a}_i \mathbb{a}_j \mathbb{a}_k$ by three non coplanar subtriangles $\mathbb{a}_i \mathbb{a}_j \mathbb{b}_{\mathbb{i}}$, $\mathbb{a}_i \mathbb{b}_{\mathbb{i}} \mathbb{a}_k$, $\mathbb{b}_{\mathbb{i}} \mathbb{a}_j \mathbb{a}_k$, meeting in one interior point $\mathbb{b}_{\mathbb{i}}$ and to associate a transversal plane $\mathcal{T}_{ik}$ to each edge $\mathbb{a}_i \mathbb{a}_k$ of the original net. The planes corresponding to the edges of one triangle $\mathbb{a}_i \mathbb{a}_j \mathbb{a}_k$ intersect in points $\mathbb{z}_{ijk}$. Now the coplanarity condition is satisfied if the transversal line of any subtriangle $\mathbb{a}_i \mathbb{a}_j \mathbb{b}_{\mathbb{i}}$ goes through $\mathbb{z}_{kpi}$ and lies in $\mathcal{T}_{ik}$.

Another construction resulting in half as many quadric segments, shape investigations and further results will be given in a longer version of this paper.

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