Analysis of C^k-subdivision surfaces at extraordinary points

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Abstract

This paper gives an analysis of surfaces generated by subdividing control nets of arbitrary topology. We assume that the underlying subdivision algorithm is stationary on the regular parts of the control nets and described by a matrix iteration around an extraordinary point. For these subdivision schemes we derive conditions on the spectrum of the matrix and its generalized eigenvectors such that surfaces are produced which are regular and k-times differentiable at their extraordinary points.

Keywords

Subdivision, extraordinary points, regular G^k-surfaces, matrix iteration.

1 Introduction

This paper gives an analysis of surfaces generated by subdividing control nets of arbitrary topology. While first order differentiability has been studied by several authors [Doo & Sabin '78], [Ball & Storry '88], [Loop '87], [Reif '95] with different detailness and completeness, no results are available, so far, on higher order smoothness except for the degree estimate by Reif [1996].

It is the intent of this paper to fill this gap. I was intrigued by the work of Ulrich Reif [1996] which helped me substantially to begin developing the ideas of this paper. Different from the aforementioned works, here, a canonical parametrization is used which eases the analysis considerably. In fact the approach taken in this paper can also be used to analyze subdivision surfaces of arbitrary dimensions. In the sequel, however, I will restrict myself to two-dimensional surfaces.

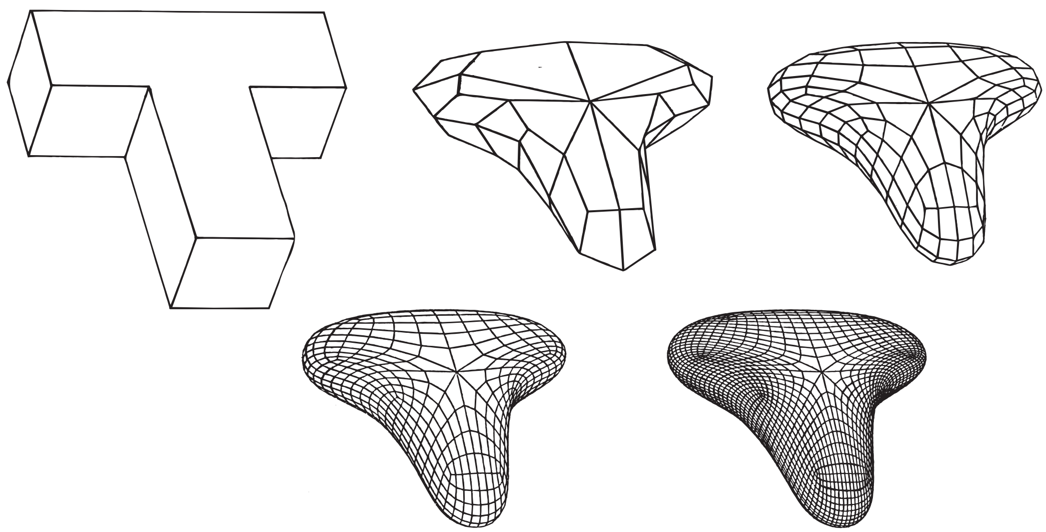

Figure 1: Subdivision by the Catmull/Clark algorithm.

The subdivision schemes covered in this paper are generalizations of local stationary schemes. Examples are the algorithms by Doo [1978], Catmull and Clark [1978], Loop [1978], and Dyn et al. [1990]. Typically, these algorithms are described as actions on a control net: They can be applied to control nets of arbitrary topology and connectedness and produce sequences of control nets such that the number of meshes is roughly quadrupled at each iteration.

Except for a fixed number of irregularities these nets consist of only triangular or only quadrilateral meshes, where each interior vertex is adjacent to exactly six of four meshes, respectively. Figure 1 shows the begin of such a sequence generated by the Catmull/Clark algorithm.

2 Describing the problem

Rather than dealing with control nets it is more advantageous for our purpose here to consider the underlying surfaces. Namely a sequence of control nets generated by one of the above mentioned algorithms defines a sequence of C^k-surfaces S_m such that every S_m contains all preceding surfaces S_j, $j < m$. Locally the limiting surface consists of quadrilateral patches $\mathfrak{q}^i_m : [0,1]^2 \to \mathbb{R}^3$, $i \in \mathbb{Z}_{3n}$, $m \in \mathbb{N}$, where $\mathfrak{q}^i_m$ belongs to S_m but not to S_{m-1}. The adjacency of these patches is shown schematically in Figure 2 for $n = 5$.

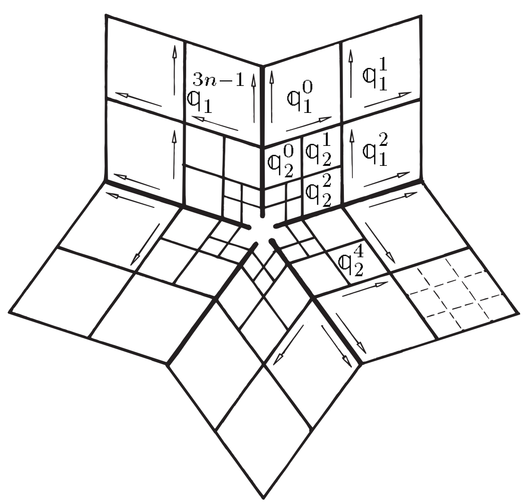

Figure 2: Subdivision surface around an extraordinary point.

Formally this means for all $i \in 3\mathbb{Z}$, all $m \in \mathbb{N}$, and all $w \in [0,1]$ that

$$\begin{aligned} \mathfrak{q}^i_{m+1}(w,1) &= \mathfrak{q}^i_m(\frac{w}{2},0) \ , \\ \mathfrak{q}^{i+1}_{m+1}(w,1) &= \mathfrak{q}^i_m(\frac{1}{2}+\frac{w}{2},0) \ , \end{aligned}$$

$$\begin{aligned} \mathfrak{q}_{m+1}^{i+1}(1,w) &= \mathfrak{q}_m^{i+2}(0,\frac{1}{2}+\frac{w}{2}) \ , \\ \mathfrak{q}_{m+1}^{i+2}(1,w) &= \mathfrak{q}_m^{i+2}(0,\frac{w}{2}) \ , \end{aligned}$$

and

$$\mathfrak{q}_m^{i}(1,w) = \mathfrak{q}_m^{i+1}(0,w) \ , \ \mathfrak{q}_m^{i+1}(w,0) = \mathfrak{q}_m^{i+2}(w,1) \ , \ \mathfrak{q}_m^{i+2}(w,0) = \mathfrak{q}_m^{i+3}(0,w) \ ,$$

where $\mathfrak{q}_m^i = \mathfrak{q}_m^{i+3n}$.

In the case of triangular nets the patches $\mathfrak{q}_m^i$ are triangular as shown in Figure 3. Note that the patches $\mathfrak{q}_m^i$ are in general macro patches which are composed of smaller triangular or quadrilateral patches as indicated for $\mathfrak{q}_1^4$ in Figures 2 and 3. Obviously, we can repartition the surface if the $\mathfrak{q}_m^i$ are triangular so as to obtain the tesselation of Figure 2. Hence we may assume in the sequel that the $\mathfrak{q}_m^i$ are quadrilateral patches.

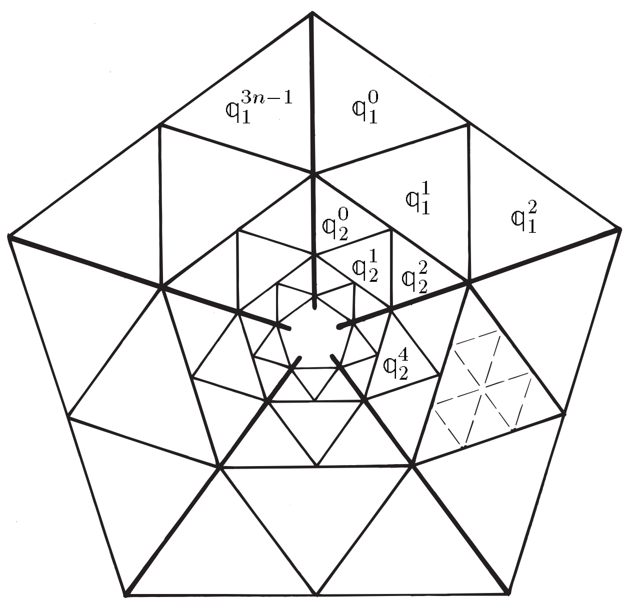

Figure 3: Subdivision surface with triangular control net around an extraordinary point.

Further, we suppose that the patches $\mathfrak{q}_m^i$ converge uniformly to a point $\mathfrak{q}_\infty$ as m tends to infinity and that the surface $\mathfrak{q}$ formed by all $\mathfrak{q}_m^i$ is k-times continuously differentiable everywhere possibly except at $\mathbb{q}_\infty$. Moreover the **surface layers** $\mathbb{q}_m : \mathbb{Z}_{3n} \times [0,1]^2 \to \mathbb{R}^3$ formed by the patches $\mathbb{q}_m(i,u,v) = \mathbb{q}^i_m(u,v)$ are defined by a certain number of control points $\mathbb{c}^m_1, \ldots, \mathbb{c}^m_s$. This means

$$\mathbb{q}_m = \sum_{j=1}^{s} B_j(i,u,v) \cdot \mathbb{c}^m_j \quad ,$$

where the B_j are the cardinal functions defined by the underlying stationary subdivision scheme. Note that the B_j need not be linearly independent. We will use a matrix notation and write

$$\mathbb{q}_m = [B_1 \ldots B_s] \begin{bmatrix} \mathbb{c}^m_1 \\ \vdots \\ \mathbb{c}^m_s \end{bmatrix} = BC_m \quad ,$$

where the $\mathbb{c}^m_j$ and $\mathbb{q}_m$ are row vectors. Moreover, throughout the entire paper $\mathbb{small\ hollow\ letters}$ will denote row vectors, while their transposes are denoted by **small boldface letters**, i.e., $\mathbf{v} = \mathbb{v}^t$.

Now we are ready to state the problem: Assuming an $s \times s$-matrix A such that $AC_m = C_{m+1}$ for all m we wish to derive conditions on A which guarantee that the layers $\mathbb{q}_m$ form a regular C^k-surface in a neighborhood of $\mathbb{q}_\infty$.

3 Solving the problem

Let $V^{-1}AV$ be the Jordan normal form of A and let $\mathbf{v}_{ij}$, $i = 1, \ldots, r$ and $j = 1, \ldots, d_i$, where $\sum d_i = s$, be the possibly complex columns of V in any arbitrary order, but such that $\mathbf{v}_{i1}, \ldots, \mathbf{v}_{id_i}$ span a (largest) irreducible invariant subspace of A. Hence this invariant subspace corresponds to a Jordan block. The associated eigenvalue is denoted by λ_i.

Moreover, we will also use the notation $\lambda = \lambda_1$ and $\mu = \lambda_2$ and assume in this section that λ and μ are real. Complex eigenvalues are discussed later in Section 4.

Further let $U = [\mathbf{v}_{11}\mathbf{v}_{21}]$ be such that the surface layer $\mathbb{x} = [x, y] = BU : \mathbb{Z}_{3n} \times [0,1]^2 \to \mathbb{R}^2$ is regular, one-to-one, and has no overlap with $\mathbb{x}_1 = BAU$ (except for the common boundary).

If $|\lambda| \geq 1$, every line segment of $\mathbb{q}_m$ parallel to the second coordinate axis would be mapped onto a line segment not shorter than the original one.

Therefore, since every sequence $\mathbb{q}_m = BA^m C_0$ is supposed to converge to a point $\mathbb{q}_\infty$, we can assume $1 > |\lambda| \geq |\mu|$. Under these conditions we can prove the following theorem, where we will use the abbreviation

$$S_\omega = \begin{cases} span\,\{y\} & \text{if } \omega = \mu \neq \lambda \\ span\,\{x^\alpha y^\beta | \alpha, \beta \in \mathbb{N}_0 \text{ and } \lambda^\alpha \mu^\beta = \omega\} & \text{otherwise} \end{cases} .$$

Theorem 1 *The layers $\mathbb{q}_m = BA^m C_0$ form a regular C^k-surface $\mathbb{q}$ for all sufficiently large m and almost all $\mathbb{q}_0$ if for all $i = 1, \ldots, r$ one of the following conditions is satisfied:*

- *either $\lambda_i = \lambda^\alpha \mu^\beta \neq -\mu$, and $|\lambda_i|$ does not lie in the possibly empty intervall $(|\mu|, |\lambda|)$, and $B\mathbf{v}_{i1} \in S_{\lambda_i}$ and $d_i = 1$*
- *or $|\lambda_i| < |\mu|^k$,*
- *or $B\mathbf{v}_{ij} = 0$ for all $j \geq 1$.*

Proof

Composing the maps $\mathbb{x}$ and $\mathbb{x}_1$ with the linear invertible map $\sigma : (\xi, \eta) \mapsto (\lambda^m \xi, \mu^m \eta)$ shows that $\mathbb{x}_m = BA^m U = \sigma \mathbb{x}$ and $\mathbb{x}_{m+1} = \sigma \mathbb{x}_1$ are also one-to-one maps without overlaps. Furthermore, $\mathbb{x}_1$ lies closer to the origin than $\mathbb{x}$. Thus all layers $\mathbb{x}_m$ together form a parametrization of some neighborhood W of the origin $\mathbb{o}$ which does not contain the origin.

We will parametrize the subdivision surface $\mathbb{q}$ over U, i.e. we will analyze the surface $\mathbb{p} : W \cup \{\mathbb{o}\} \to \mathbb{R}^3$ defined piecewise by

$$\mathbb{p}(\xi, \eta) = \begin{cases} \mathbb{q}_m(u, v) \ , & \text{if } (\xi, \eta) = \mathbb{x}_m(u, v) = (\lambda^m x, \mu^m y) \ , \\ \mathbb{q}_\infty \ , & \text{if } (\xi, \eta) = (0, 0) \ . \end{cases}$$

Now we show that the coordinates of $\mathbb{p}$ are differentiable: Each coordinate q_0 of $\mathbb{q}_0$ can be written as

$$q_0 = \sum_i a_i + b$$

where the sum $\sum a_i$ extends over all i satisfying $|\lambda_i| \geq |\mu^k|$ and where the a_i are of the form $a_i = \sum q_{\alpha\beta} x^\alpha y^\beta$, $\lambda^\alpha \mu^\beta = \lambda_i$, and where $b \in span\,\{B\mathbb{v}_{ij} | j = 1, \ldots, d_i$, where $|\lambda_i| < |\mu^k|\}$. Thus it suffices to consider a component a_i and the component b:

(1) Let $q_0 = a_i$, then

$$\begin{aligned} q_m &= \lambda_i^m \sum q_{\alpha\beta} x^\alpha y^\beta \\ &= \sum q_{\alpha\beta} (\lambda^m x)^\alpha (\mu^m y)^\beta \ . \end{aligned}$$

Thus in this case $p(\xi, \eta) = \sum q_{\alpha\beta} \xi^\alpha \eta^\beta$ is a polynomial.

(2) Let $q_0 = b$, then we have $\|q_m\| = o(|\mu|^{km})$ and thus

$$\left\| \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\beta}{\partial \eta^\beta} (q_m \circ \mathbb{x}^{-1}) \right\| = o(|\mu|^{km}) \ , \quad \alpha, \beta \in \mathbb{N}_0 \ ,$$

since composition with $\mathbb{x}^{-1}$ and differentiation are linear operators which do not slow down the contraction rate of a sequence.

Now on using $\mathbf{x}_m^{-1}(\xi, \eta) = \mathbb{x}^{-1}(\lambda^{-m}\xi, \mu^{-m}\eta)$, we obtain for $\alpha + \beta \leq k$

$$\begin{aligned} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\beta}{\partial \eta^\beta} \ p(\xi, \eta) \right| &= o(|\lambda|^{-\alpha m} |\mu|^{(k-\beta)m}) \\ &= o(|\mu|^{(k-\alpha-\beta)m}) \qquad (1) \\ &= o(\|(\xi, \eta)\|^{(k-\alpha-\beta)}) \ . \end{aligned}$$

Hence if $q_0 = b$, all derivatives of $p(\xi, \eta)$ up to order k converge to 0 as $(\xi, \eta) \to \mathbb{o}$ which shows that $\mathbb{p}$ is k-times differentiable.

In order to prove that $\mathbb{p}$ is regular, we write

$$\mathbb{q}_0 = \mathbb{a} + \mathbb{b}x + \mathbb{c}y + BR$$

where $\|BA^m R\| = o(|\mu|^m)$. Then as shown above we have

$$\mathbb{p}(\xi, \eta) = \mathbb{a} + \mathbb{b}\xi + \mathbb{c}\eta + \mathbb{r}(\xi, \eta) \ , \quad \|\mathbb{r}\| = o(\|(\xi, \eta)\|) \ ,$$

which is regular for small ξ and η if $\mathbb{b}$ and $\mathbb{c}$ are linearly independent, i.e. $\mathbb{q}$ formed by the layers $\mathbb{q}_m$, where m is sufficiently large, is regular for almost all $\mathbb{q}_0$. ■

The proof of the theorem above gives more than stated. Namely the estimates in 1 still apply if $|\lambda|^{-\alpha}|\mu|^{k-\beta} \leq \mu^0$, i.e. if $\beta \leq k - c\alpha$, where $c = \dfrac{\log|\lambda|}{\log|\mu|}$, see Figure 4.

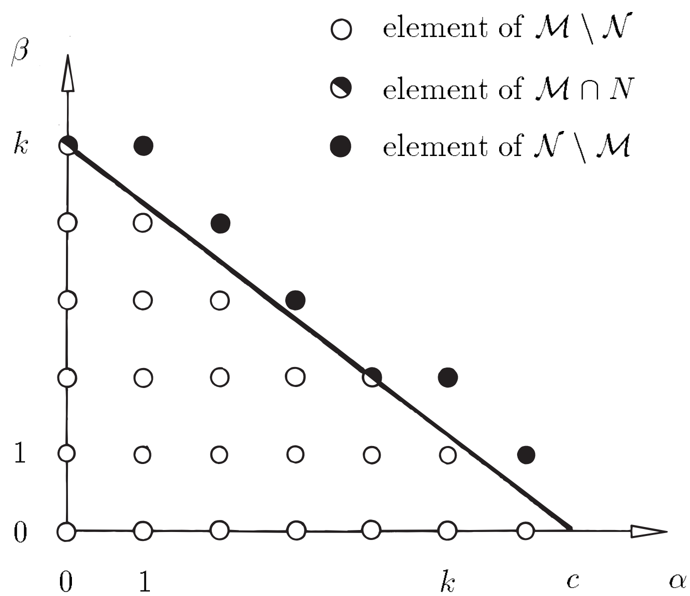

Figure 4: The sets $\mathcal{M}$ and $\mathcal{N}$.

Thus the mixed partial derivatives

$$D^{\alpha\beta} p = \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\beta}{\partial \eta^\beta} \; p$$

exist continuously for all $(\alpha, \beta) \in \mathcal{M} := \{(\alpha, \beta) | \alpha, \beta \in \mathbb{N}_0 \text{ and } |\lambda^\alpha \mu^\beta| \geq |\mu^k|\}$.

In this general form one can also show a converse of the theorem:

Proposition 1 *Let $\mathcal{N} := \{(\alpha, \beta) | \alpha, \beta \in \mathbb{N}_0$ and $\mu^k \geq |\lambda^\alpha \mu^\beta| > \mu^{k+1}\}$. Then if all derivatives $D^{\alpha\beta}\mathbb{p}$, $(\alpha, \beta) \in \mathcal{M} \cup \mathcal{N}$, exist continuously for all $\mathbb{q}_0$, we have for all $i = 1, \ldots, r$ one of the following conditions:*

- *$B\mathbf{v}_{i1} \in S_{\lambda_i}$ where $\lambda_i = \lambda^\alpha \mu^\beta$ and $B\mathbf{v}_{ij} = 0$ for all $j \geq 2$,*
- *or $|\lambda_i| < |\mu|^k$,*
- *or $B\mathbf{v}_{ij} = 0$ for all $j \geq 1$.*

Proof
The Taylor expansion of $\mathbb{p}(\xi, \eta)$ around $(\eta, 0)$ reads

$$\mathbb{p}(\xi, \eta) = \sum_{j=0}^{k} \frac{1}{j!} D^{0j} \mathbb{p}(\xi, 0)\eta^j + \mathbf{o}(|\eta|^k)$$

where $\mathbb{o}(|\eta|^k)$ stands for a term $\mathbb{r}(\eta)$ with $\|\mathbf{r}(\eta)\| = o(|\eta|^k)$. Using Taylor expansions again, we obtain a sum of the form

$$\mathbb{p}(\xi, \eta) = \sum_{\mathcal{M}\cup\mathcal{N}} \xi^\alpha \eta^\beta \mathbb{d}_{\alpha\beta} + \sum_{\mathcal{N}} \mathbb{o}(|\xi^\alpha \eta^\beta|) \ .$$

Thus we have

$$BA^m C_0 = \mathbb{p}(\lambda^m x, \mu^m y) = \sum_{\mathcal{M}\cup\mathcal{N}} (\lambda^\alpha \mu^\beta)^m x^\alpha y^\beta \mathbb{d}_{\alpha\beta} + \mathbb{o}(|\mu|^{km}) \ .$$

This asymptotic behaviour can only be observed for all $\mathbb{q}_0$ under the claimed spectral properties. ■

4 Complex eigenvalues

It is rather straightforward to extend the above analysis to the case where λ and μ, $\mathbb{v}_{11}$ and $\mathbb{v}_{21}$, and x and y are complex conjugate pairs. We use the invertible transformation $T : \mathbb{C} \to \mathbb{R}^2$, $z \to (\mathrm{Re}\ z, \mathrm{Im}\ z)$ and assume that the surface layer $Tx = TB\mathbf{v}_{11} : \mathbb{Z}_{3n} \times [0,1]^2 \to \mathbb{R}^2$ is one-to-one and has no overlap with $TBA\mathbf{v}_{11} = T\lambda x$. Then on using $S_\omega = span\{x^\alpha \bar{x}^\beta | \alpha, \beta \in \mathbb{N}_0$ and $\lambda^\alpha \bar{\lambda}^\beta = \omega\}$ Theorem 1 is still valid:

Proof
We can use the proof of Theorem 1 with slight modifications: Obviously, $T\lambda^m x$ and $T\lambda^{m+1}x$ are also one-to-one maps without overlaps. Furthermore, $T\lambda x$ lies closer to the origin $\mathbb{o}$ than Tx. Thus all layers $T\lambda^m x$ together form a parametrization of some neighborhood W of the origin which does not contain the origin. Again we will parametrize the subdivision surface $\mathbb{q}$ over W, i.e. we will analyze the surface $\mathbb{p} : W \cup \{\mathbb{o}\} \to \mathbb{R}^3$ defined piecewise by

$$\mathbb{p}(\xi, \eta) = \begin{cases} \mathbb{q}_m(u, v) \ , & \text{if } \xi + i\eta = \lambda^m x \ , \\ \mathbb{q}_\infty = \lim\limits_{m\to\infty} \mathbb{q}_m \ , & \text{if } \xi + i\eta = 0 \ . \end{cases}$$

Now we show that the coordinates of $\mathbb{p}$ are differentiable: Each coordinate q_0 of $\mathbb{q}_0$ can be written as

$$q_0 = \sum_i a_i + b \ ,$$

where b lies in the invariant subspace of A associated with all λ_i such that $|\lambda_i| < |\lambda|^k$, and $a_i = \sum q_{\alpha\beta} x^\alpha \bar{x}^\beta$, where all α, β in this sum satisfy $\lambda^\alpha \bar{\lambda}^\beta = \lambda_i$, and $a_i = \bar{a}_j$ for $\lambda_i = \bar{\lambda}_j$. Thus it suffices to consider the following two cases:

(1) Let $q_0 = a_i + \bar{a}_i$, then we get

$$\begin{aligned} q_m &= \lambda_i^m \sum q_{\alpha\beta} x^\alpha \bar{x}^\beta + \bar{\lambda}_i^m \sum \bar{q}_{\alpha\beta} \bar{x}^\alpha x^\beta \\ &= \sum q_{\alpha\beta} (\lambda^m x)^\alpha (\bar{\lambda}^m \bar{x})^\beta + \sum \bar{q}_{\alpha\beta} (\bar{\lambda}^m \bar{x})^\alpha (\lambda^m x)^\beta \\ &= \text{ a polynomial in } \xi \text{ and } \eta \text{ with real coefficients.} \end{aligned}$$

(2) Let if $q_0 = b$, then we have $\|q_m\| = o(|\lambda|^{km})$ and thus

$$\|D^{\alpha\beta}(q_m \circ (Tx)^{-1})\| = o(|\lambda|^{km})$$

since composition with $(Tx)^{-1}$ and differentiation are linear operators, which do not slow down the contraction rate of a sequence.

Now on using the rotation $R(\xi, \eta) = (\xi \cos\phi - \eta \sin\phi, \eta\cos\phi + \xi\sin\phi)$ by the angle $\phi = \text{arc } \lambda^{-m}$ we obtain

$$(T\lambda^m x)^{-1}(\xi, \eta) = (Tx)^{-1}(|\lambda|^{-m} R(\xi, \eta))$$

and thus

$$\begin{aligned} D^{\alpha\beta} p(\xi, \eta) &= D^{\alpha\beta}(q_m \circ (T\lambda^m x)^{-1}) \\ &= |\lambda|^{-m(\alpha+\beta)} \sum_{\gamma+\delta=\alpha+\beta} d_{\gamma\delta} D^{\gamma\delta}(q_m \circ (Tx)^{-1}) \end{aligned}$$

where all $|d_{\gamma\delta}|$ depend on ϕ and are bounded. Hence we have

$$|D^{\alpha\beta} p(\xi, \eta)| = o(\|(\xi, \eta)\|^{(k-\alpha-\beta)})$$

as before. To prove that $\mathbb{p}$ is regular around $\mathbb{q}_\infty$ we write

$$\mathbb{q}_0 = \mathbb{a} + \mathbb{b}x + \overline{\mathbb{b}}\overline{x} + BR \ ,$$

where $\mathbb{a} \in \mathbb{R}^3, \mathbb{b} \in \mathbb{C}^3$, and $||BA^m R|| = o(|\mu|^i)$. Then it follows as above that

$$\mathbb{p}(\xi, \eta) = \mathbb{a} + \mathbb{b}\xi + \mathbb{c}\eta + \mathbb{r}(\xi, \eta) \ ,$$

where $||\mathbf{r}|| = o(||(\xi, \eta)||)$. This map is regular for small (ξ, η) provided that $\mathbf{b}$ and $\overline{\mathbf{b}}$ are linearly independent. This completes the proof. ■

Since $|\lambda| = |\bar{\lambda}|$, it is even simpler than in Section 3 to state and to prove the converse fact:

Proposition 2 *If $\mathbb{p}$ as defined in the last proof is k-times continuously differentiable for all $\mathbb{q}_0$, then A has the spectral properties required in Theorem 1.*

Proof
Let $\mathbb{a}(\xi, \eta)$ be the Taylor polynomial of degree k of $\mathbb{p}$ around $\mathbb{o}$. Then we have

$$\mathbb{p}(\xi, \eta) = \mathbb{a}(\xi, \eta) + \mathbb{o}(\|(\xi, \eta)\|^k)$$

and for $(\xi, \eta) = \lambda^m x$

$$BA^m C_0 = \mathbb{a}(\xi, \eta) + \mathbb{o}(\|(\xi, \eta)\|^k)$$

which can be written as

$$\begin{aligned} BA^m C_o &= \sum_{\alpha+\beta \le k} \mathbb{c}_{\alpha\beta} (\lambda^m x)^\alpha (\bar{\lambda}^m \bar{x})^\beta + \mathbb{o}(\|(\xi, \eta)\|^k) \\ &= \sum \mathbb{c}_{\alpha\beta} (\lambda^\alpha \bar{\lambda}^\beta)^m x^\alpha \bar{x}^\beta + \mathbb{o}(\|(\xi, \eta)\|^k) \ , \end{aligned}$$

where $\mathbb{c}_{\alpha\beta} = \bar{\mathbb{c}}_{\beta\alpha}$. This asymptotic behaviour can only be observed for all $\mathbb{q}_0$ under the claimed spectral properties. ■

5 Concluding remark

In a forthcoming paper we will show how one can construct subdivision algorithms for the generation of C^k-surfaces. These surfaces are piecewise polynomial of bidegree $(rk+r, rk+r)$ where $r \geq 0$ can be arbitrarily chosen. The number r denotes the total degree of the surface around an extraordinary point if the surface is viewed as a function over the tangent plane.

In another forthcoming paper together with U. Reif we will show that the above degree $(rk + r, rk + r)$ is in general best possible.

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