Circle and Sphere as rational splines

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Abstract
A simple method is given to construct periodic spline representations for circles. These are n-times differentiable and of minimal degree. Further, the extension to spheres is discussed.
Keywords circle, sphere, stereographic projection, periodic splines

1. INTRODUCTION

The simplest geometric objects after lines are circles. Their simplicity and symmetry accounts for their importance in geometric modelling. For example, circles come up naturally in surfaces of revolution.

Since CAD-systems are often based on rational splines, i.e. piecewise rational polynomials, to represent curves and surfaces, it is interesting and often necessary to represent circles and spheres in this way. Already Apollonios (200 B.C.) knew quadratic parametrizations of the circle and Hipparchos (180-125 B.C.) used the stereographic projection for the sphere, cf. (Blaschke, 1954, pp 49&142). A complete characterization of all rational patches on the sphere was first given, however, in (Dietz et al., 1993).

Strikingly only few attempts have been made so far to smoothly parametrize the full circle. (Piegl & Tiller, 1989) describe a piecewise quadratic parametrization and show that it is not possible to represent the full circle by quadratic C^1-B-splines.

Chou (Chou, 1995) constructs a representation of the entire circle by one quartic polynomial segment and observes that all weights of the quintic Bézier representation are positive. The corresponding periodic homogeneous B-spline representation is also not differentiable.

Therefore and in consideration of the fact that many numerical methods used in geometric modelling need differentiable parametrizations, we will present a simple method to obtain $(n-1)$-times differentiable periodic B-spline representations for the circle of degree $2n$ and show that this degree is minimal. As an example we derive explicitly a periodic representation by quartic C^1-B-splines.

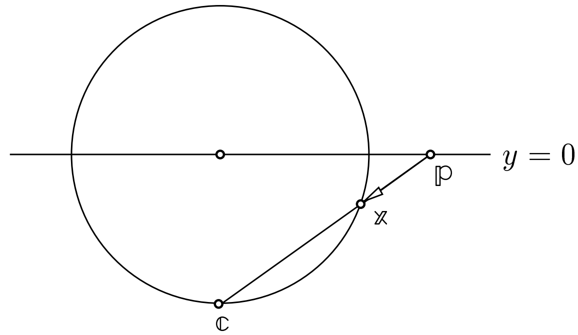

Figure 1: Stereographic projection.

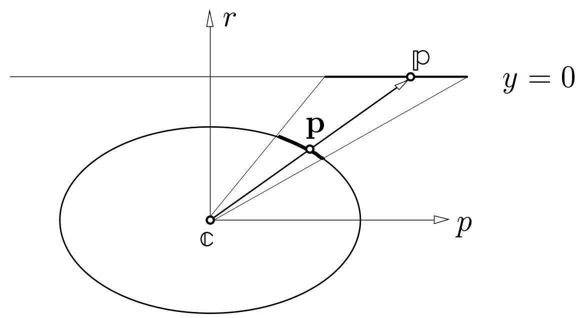

Figure 2: Central projection.

Furthermore, we will discuss representations of entire spheres.

2. THE CONSTRUCTION

Throughout the paper we consider the unit circle given by a periodic parametrization $\mathbb{x}(t)$. Small hollow letters denote homogeneous coordinate columns. Thus for $\mathbb{x} = [x \quad y \quad z]^{\mathrm{t}}$ we have

$$x^2 + y^2 = z^2 \ ,$$

where z is the homogenizing coordinate. For simplicity, we will always assume that z is positive and that x, y, z have no common divisor.

Any parametrization $\mathbb{x}(t)$ of the circle can be obtained from a parametrization $\mathbb{p}(t) = [p(t) \quad 0 \quad r(t)]^{\mathrm{t}}$ of the line $y = 0$ by the stereographic projection

$$\mathbb{x} = \begin{bmatrix} 2pr \\ r^2 - p^2 \\ p^2 + r^2 \end{bmatrix}$$

with center $\mathbb{c} = [0 \quad -1 \quad 1]^{\mathrm{t}}$ as illustrated in Figure 1.

Hence we need a periodic parametrization of the projective line $y = 0$. Any such parametrization can be obtained from a closed curve around $\mathbb{c}$ by a central projection with center $\mathbb{c}$ as illustrated in Figure 2.

For notational convenience we will use a second coordinate system, namely the affine system with origin $\mathbb{c}$ and unit vectors $[1 \quad 0 \quad 1]^{\mathrm{t}}$ and $[0 \quad 1 \quad 0]^{\mathrm{t}}$. The respective coordinate columns are denoted by small bold letters. Thus $\mathbb{p}$ and $[p/r \quad 1]^{\mathrm{t}}$ represent the same point and the points represented by $\mathbb{c}, \mathbb{p}$, and $\mathbf{p} = [p \quad r]^{\mathrm{t}}$ lie on one line.

Note that if $\mathbf{p}(t)$ orbits around $\mathbb{c}$ once, then $\mathbb{p}(t)$ and its stereographic projection $\mathbb{x}(t)$ trace out the line and circle twice. If $\mathbf{p}(t)$ is centrally symmetric, then both cycles of the line and circle are parametrized alike. However, note for later reference that the coordinate representation $\mathbb{p}$ has different signs in both cycles.

3. AN EXAMPLE

Here we apply the construction above to derive an explicit representation of the circle by a differentiable periodic spline of degree 4. Let $\alpha = 90°/m$, $m \geq 2$, and

$$\mathbf{p}_i = \begin{bmatrix} \cos(3\alpha + 2i\alpha) \\ \sin(3\alpha + 2i\alpha) \end{bmatrix}$$

and let $N_i^2(t)$ denote the piecewise quadratic B-spline over the knots i, $i+1$, $i+2$, $i+3$. Then

$$\mathbf{p}(t) = \sum_{i \in \mathbb{Z}} \mathbf{p}_i N_i^2(t)$$

is a centrally symmetric spline with $\mathbf{p}(t) = -\mathbf{p}(t+m)$. It is shown in Figure 3 for $m = 3$.

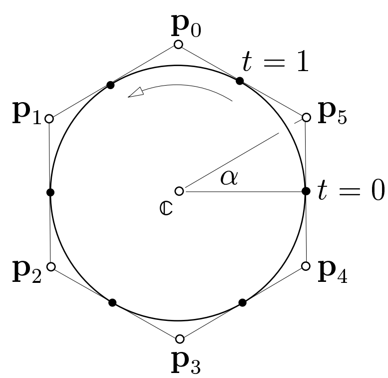

Figure 3: Centrally symmetric spline.

The corresponding parametrization $\mathbb{x}$ of the circle is a piecewise quartic C^1-spline given by

$$\mathbb{x}(t) = \sum \mathbb{x}_i N_i^4(t)$$

where N_i^4 denotes the B-splines over the knots $\lfloor i/3 \rfloor$, $\lfloor (i+1)/3 \rfloor$, ..., $\lfloor (i+5)/3 \rfloor$, $\lfloor x \rfloor := \max\{i \leq x | i \in \mathbb{Z}\}$, and where

$$\begin{aligned}
\mathbb{x}_{3l\pm1} &= \begin{bmatrix} c\cos(\beta + 4l\alpha \pm \alpha) \\ c\sin(\beta + 4l\alpha \pm \alpha) \\ 1 \end{bmatrix} \\
\mathbb{x}_{3l} &= \begin{bmatrix} d\cos(\beta + 4l\alpha) \\ d\sin(\beta + 4l\alpha) \\ \omega \end{bmatrix} \\
\beta &= 2\alpha - 90° \\
c &= 1/\cos\alpha \\
d &= (\cos^2\alpha + 2)/(3\cos^2\alpha) \\
\omega &= (2\cos^4\alpha - \cos^2\alpha + 2)/(3\cos^2\alpha)
\end{aligned}$$

These control points $\mathbb{x}_i$ are shown in Figure 4.

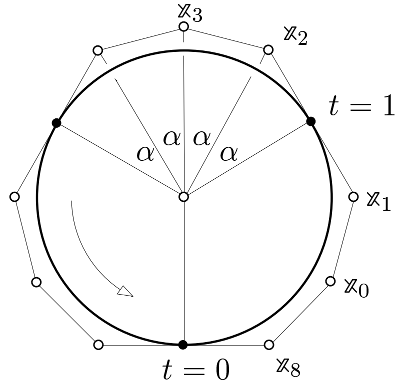

Figure 4: The circle as periodic spline.

Note that $\mathbb{x}(t)$ is symmetric with respect to rotations around the angle 4α. In Section 5 we will show more generally that rotational symmetry of $\mathbf{p}$ implies rotational symmetry of $\mathbb{x}$ with respect to the double angle. Before we come to it, we show that a C^{n-1}-spline representation of the circle must be of degree $2n$ at least. Thus the construction given in Section 2 gives minimal degree representations of the circle if $\mathbf{p}$ is a C^{n-1}-spline of degree n.

Remark 1 *For $\alpha = 90°$ $(m = 1)$ it is not possible to represent $\mathbb{x}$ as a C^1-spline. However, one can represent $\mathbb{x}$ by one quartic polynomial. The Bézier representation has zero weights (Chou, 1995).*

Example 1 *Figure 5 shows a hyperboloid of one sheet and the corresponding control net of a tensor product B-spline representation of degree $(2, 4)$.*

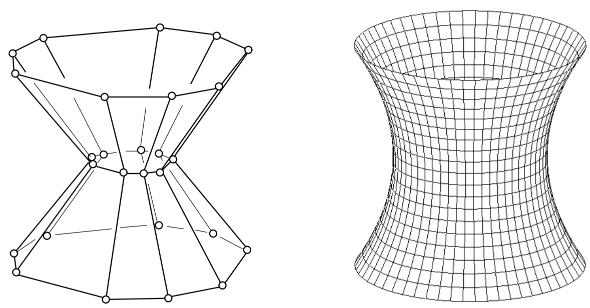

Figure 5: Hyperboloid of one sheet and corresponding control net.

4. POSITIVE WEIGHTS

In this section we will consider Bézier representations of the circle.

First observe that the Bézier points of the parametrization in Section 3 are given by

$$\frac{1}{2}(\mathbb{x}_1 + \mathbb{x}_2),\ \mathbb{x}_2,\ \mathbb{x}_3,\ \mathbb{x}_4,\ \frac{1}{2}(\mathbb{x}_4 + \mathbb{x}_5),\quad \text{etc.}\ ,$$

thus the weights i.e. z-coordinates are all positive.

The Bézier-representation of an arbitrary circle segment

$$\mathbb{x}^l(t) = \sum_{i=0}^{2k+2} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} B_i^{2k+2}(t)$$

can be obtained from the Bézier representation of the corresponding segment $\mathbf{p}^l$ of the preimage $\mathbf{p}$:

$$\mathbf{p}^l(t) = \sum_{i=0}^{k+1} \begin{bmatrix} p_i \\ r_i \end{bmatrix} B_i^{k+1}(t)$$

by the product formula for Bernstein polynomials (Farin, 1990, p 65).

In particular we obtain for the z-coordinate:

$$\binom{2k+2}{j} z_j = \sum_{i=\max(0,j-k-1)}^{\min(k+1,j)} (p_i p_{j-i} + r_i r_{j-i}) \binom{k+1}{i} \binom{k+1}{j-i} \ .$$

If $\mathbb{x}^l$ represents at most a semicircle we can because of the symmetry properties shown in Section 8 assume without loss of generality that $\mathbf{p}^l$ lies entirely in the first quadrant of the p, r-plane, i.e.

$$\begin{aligned} 0 &\leq p_0, r_n \\ 0 &< p_{i+1}, r_i \quad \text{for } i = 0 \dots k \ . \end{aligned}$$

Thus the positivity of the weights z_i follows immediately.

5. CONTINUITY ORDER OF THE PREIMAGE CURVE p

In Section 2 we showed that any periodic parametrization $\mathbb{x}(t)$ of the unit circle is related by the stereographic projection to a closed curve $\mathbf{p}(t)$ which is centrally symmetric to the center of projection $\mathbb{c}$. Here we will show that $\mathbf{p}(t)$ and $\mathbb{x}(t)$ are of the same continuity order.

Consider the inverse projection from the line $y = 0$ onto the circle. It is given by

$$2p \cdot \mathbb{p} = \begin{bmatrix} z-y \\ 0 \\ x \end{bmatrix} \quad \text{and also by} \quad 2r \cdot \mathbb{p} = \begin{bmatrix} x \\ 0 \\ z+y \end{bmatrix} \ .$$

Since z is assumed to be positive, $z - y$ and $z + y$ are non-negative. Thus we have

$$p = \pm\sqrt{(z-y)/2} \quad \text{and} \quad r = x/2p$$

and also

$$r = x/2r \quad \text{and} \quad r = \pm\sqrt{(z+y)/2} \ ,$$

where the signs of $p(\mathbb{x})$ and $r(\mathbb{x})$ change at $\mathbb{x} = [0 \quad 1 \quad 1]^{\mathrm{t}}$ and $\mathbb{x} = \mathbb{c}$, respectively, as already noted in Section 2.

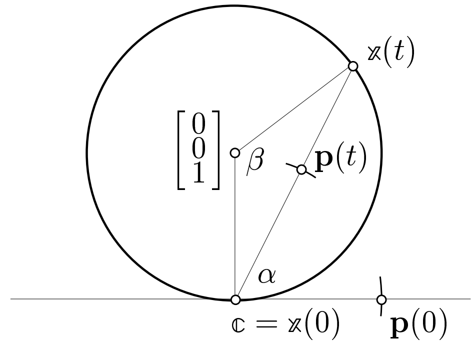

Figure 6: Peripheral and perimeter angle.

Since the first square root does not change its sign when the other does and vice versa, the four expressions above imply that $\mathbf{p}$ is of the same continuity order as $\mathbb{x}$.

6. MINIMAL DEGREE

Let x, y, z be piecewise polynomial without common divisor. Then $z - y$ and $z + y$ also have no common divisor. Therefore and because of $(z-y)(z+y) = x^2$ one can factor $x = 2pr$ such that $z - y = 2p^2$ and $z + y = 2r^2$, cf. (Kubota, 1972). Hence, $\mathbf{p}$ is also piecewise polynomial, while the degree of $\mathbb{x}$ is twice as high as that of $\mathbf{p}$.

Since polynomials are not periodic, $\mathbf{p}$ must be piecewise polynomial of degree $k+1$. Hence a k-times differentiable parametrization $\mathbb{x}(t)$ of the circle must be at least of degree $2k + 2$.

7. MINIMAL NUMBER OF SEGMENTS

The B-spline control polygon of a periodic, centrally symmetric curve $\mathbf{p}$ consists of one or an even number of control points. Thus a non-degenerate $\mathbf{p}$ must have at last 4 control points.

If $\mathbf{p}$ is of degree n and continuity order $(n - 1)$, then the number of its B-spline control points equals the number of its segments. Therefore $\mathbf{p}$ has at least 4 segments while $\mathbb{x}$ is of degree $2n$ and has at least 2 segments. Consequently a periodic C^k-parametrization $\mathbb{x}(t)$ of the circle must consist of at least 2 segments to be of minimal degree $2k + 2$.

8. SYMMETRIC REPRESENTATIONS

In order to investigate symmetry properties of $\mathbb{x}(t)$ we can assume without loss of generality that $\mathbb{x}(0) = \mathbb{c}$. Then we have $r(0) = 0$.

Now let $\alpha(t)$ be the angle $\mathbf{p}(0), \mathbb{c}, \mathbf{p}(t)$ and $\beta(t)$ be the angle $\mathbb{x}(0), [0 \quad 0 \quad 1]^{\mathrm{t}}, \mathbb{x}(t)$, see Figure 6. Recall that the peripheral angle α is half the perimeter angle β.

Using these angles we can write

$$\mathbb{x}(t) = \xi(\beta) \begin{bmatrix} -\sin\beta \\ \cos\beta \\ 1 \end{bmatrix} \quad , \quad \mathbf{p}(t) = \rho(\alpha) \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \quad .$$

The formula of the stereographic projection in Section 2 then shows that

$$\xi(\beta) = \xi(2\alpha) = p^2 + r^2 = \rho^2(\alpha) \ .$$

If $\mathbf{p}(t)$ is invariant under a rotation around $\mathbb{c}$ by an angle ϕ, then

$$\rho(\alpha) = \rho(\alpha + \phi)$$

which implies

$$\xi(\beta) = \xi(\beta + 2\phi) \ ,$$

i.e. $\mathbb{x}$ is invariant under a rotation around the origin by 2ϕ.

Further, if $\mathbf{p}$ is symmetric with respect to the line through $\mathbb{c}$ and $\mathbf{p}(t)$, then

$$\rho(\alpha + \delta) = \rho(\alpha - \delta) \ , \quad \alpha = \alpha(t)$$

which implies

$$\xi(\beta + 2\delta) = \xi(\beta - 2\delta) \ , \quad \beta = \beta(t) \ .$$

Consequently $\mathbb{x}$ is symmetric with respect to the line through the origin $[0 \quad 0 \quad 1]^{\mathrm{t}}$ and $\mathbb{x}(t)$.

9. CONTINUITY OF ODD DERIVATIVES

If a homogeneous representation $\mathbb{x}(t)$ of the circle is k-times differentiable, then the affine coordinate functions x/z and y/z are also k-times differentiable. However, x/z and y/z can be smoother. In particular we show that $2k$-times differentiable symmetric affine circle representations are also $2k+1$-times differentiable. This generalizes the result by (Piegl & Tiller, 1989) for $n = 0$.

Let $\mathbf{x}(t)$ be a symmetric parametrization of the unit circle where $\mathbf{x} = [x \quad y]^{\mathrm{t}}$ is an affine coordinate column. Without loss of generality we assume $\mathbf{x}(0) = [0 \quad -1]^{\mathrm{t}}$ and symmetry around $t = 0$, i.e. we have

$$\mathbf{x}(t) = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \mathbf{x}(-t) \ .$$

Further suppose that $\mathbf{x}(t)$ is $2k$-times differentiable and that the left and right hand side derivatives of order $n = 2k+1$ of $\mathbf{x}(t)$ exist at $t = 0$. Due to the symmetry property these derivatives are related by

$$\mathbf{x}^{(n)}(0+) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathbf{x}^{(n)}(0-) \ ,$$

which implies identity of the first coordinates.

Recall that the arclength of $\mathbf{x}(t)$ is given by

$$s = \int \|\dot{\mathbf{x}}\| dt \ .$$

Hence $s(t)$ is as often differentiable as $\mathbf{x}(t)$. Now we differentiate

$$\mathbf{x}(t) = \mathbf{x}(s(t)) = \begin{bmatrix} -\sin(s) \\ \cos(s) \end{bmatrix}$$

by means of the chain and product rule and obtain

$$\mathbf{x}^{(n)}(0+) - \mathbf{x}^{(n)}(0-) = (s^{(n)}(0+) - s^{(n)}(0-)) \frac{d}{ds}\mathbf{x}(0) \ .$$

Since $\frac{d}{ds}\mathbf{x}(0) = \pm [1 \quad 0]^{\mathrm{t}}$ the equation above implies the identity of the second coordinates of the n-th derivatives.

Hence $\mathbf{x}(t)$ is n-times differentiable at $t = 0$.

10. THE SPHERE

The construction in Section 2 cannot be extended to spheres for the following reasons. The stereographic projection does not establish a one-to-one correspondence between a sphere and a plane and, moreover, it is impossible to map a planar domain differentiably onto the entire sphere without singularities, see e.g. (Prautzsch & Trump, 1992).

But it is possible to decompose the sphere into several triangular or quadrilateral patches. Farin, Piper, and Worsey (1987) represent the octants of a sphere and Dietz (1995) showed that for arbitrarily prescribed circular boundary curves there exists a tensor product patch of degree $(2, 4)$ on the sphere. Thus with the cube in mind one sees that it is possible to describe the entire sphere with 6 tensor product patches of degree $(2, 4)$.

We will show that lower degree representations of the entire sphere are not possible: Assume, that there exist f triangular quadratic regular patches covering the sphere with altogether v vertices. Then recall from (Dietz, 1995, Thm. 3.4) that the angles of each patch sum to $180°$. Thus we have

$$v \cdot 360° = \text{sum of all angles} = f \cdot 180°$$

and consequently $v = f/2$. Since there are $e = 3f/2$ many boundary curves, Euler's identity $v - e + f = 2$ is not satisfied which proves the assumption wrong.

It is also impossible to partition the entire sphere into biquadratic tensorproduct patches. Namely, every such patch could be decomposed into two quadratic triangular patches.

Since every irreducible parametrization of the sphere is of even degree (Dietz et al., 1993), the sphere cannot be decomposed into triangular or quadrilateral patches whose degree is less than 4 or $(2, 4)$, respectively.

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