Triangular G^k Splines

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A general technique to build piecewise polynomial G^k surfaces of degree $\mathcal{O}(k)$ is well-known. However, explicit constructions of regular parametrizations for arbitrarily high smoothness order k have not yet been presented for triangular patches. This is done in this paper, where we introduce box and half-box spline surfaces with multiple extraordinary control points.

Categories and Subject Descriptors: I.3.5 [**Computer graphics**]: Surface representation, splines

General Terms: Algorithms.

Additional Key Words and Phrases: Geometric Modelling, CAD, Curves and Surfaces.

1. INTRODUCTION

Smooth free form surfaces are commonly built from polynomial patches. In particular, subdivision provides a powerful method to generate free form surfaces. A subdivision surface is defined as the limiting surface of a mesh sequence. In general, subdivision surfaces need not be polynomial as for example the butterfly- and $\sqrt{3}$-subdivision surfaces [Dyn et al. 1990; Kobbelt 2000], but many well-known subdivision algorithms are derived from regular box spline subdivision algorithms. Their limiting surfaces consist of infinitely many polynomial patches. For example, the midpoint schemes [Prautzsch 1998; Zorin and Schröder 2001] which include the Doo-Sabin, the Catmull-Clark and the Qu-Algorithm [Doo and Sabin 1978; Catmull and Clark 1978; Qu 1990] are based on the Lane-Riesenfeld algorithm [Lane and Riesenfeld 1980] for uniform tensor-product splines. Loop's algorithm [Loop 1987] is based on the subdivision algorithm for quartic box splines [Prautzsch 1984].

Box spline based subdivision can also be understood as a process by which more and more polynomial patches are added to an initial box spline surface defined by a mesh being subdivided. The initial surface consists of finitely many patches and has holes associated with the irregularities in the mesh. Under subdivision such a hole is filled with infinitely many patches surrounding a so-called extraordinary point.

While subdivision is elegant and simple, subdivision surfaces typically suffer from shape artifacts [Karciauskas et al. 2004; Peters and Reif 2004] and it has been shown [Reif 1996; Prautzsch and Reif 1999] that generating smoother subdivision surfaces

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with second or higher order smoothness at the extraordinary points cannot be as simple and elegant. Therefore, other methods are preferred to fill n-sided holes in piecewise polynomial spline surfaces.

The construction by Hahn [Hahn 1989] is one of the oldest. There, the surfaces are piecewise polynomial of degree $\mathcal{O}(k^2)$. More recently, Reif presented singularly parametrized G^k surfaces with polynomial degree $2k+2$, see [Reif 1998]. Simultaneously, the same technique was used to construct regular G^k surfaces of the same degree $2k+2$ in [Prautzsch 1997]. Further improvements were made in [Peters 2002].

So far, the construction in [Prautzsch 1997] and also in [Peters 2002] has been outlined only for $k=2$. In this paper, we show that these ideas can be extended to construct hole fillings for three direction box and half-box splines of any smoothness order k. The polynomial degree of our fillings is $\max\{4k+1, \lceil \frac{3}{2}k+1\rceil r\}$, where $r \in \mathbb{N}$ can be chosen arbitrarily. The number r controls the flexibility at the extraordinary point, i.e. the filling consists of a reparametrized, split and modified polynomial of degree r. A crucial point is the construction of a parametrization for the filling polynomial. We present two different parametrizations. The first is singular in analogy to the parametrization for quadrilateral patches in [Reif 1998]. We show that this singular parametrization is a special degenerate parametrization from a class of regular parametrizations we present second.

2. BOX SPLINE SURFACES

The symmetric box splines of order m over the triangular grid spanned by $[1\ 0]^t$, $[0\ 1]^t$, $[-1\ \ -1]^t$ are C^{2m} continuous, and they are piecewise polynomial of degree $3m+1$ on each triangle of the grid. The grid is shown in Figure 1. We are using only these box splines since we need their symmetries.

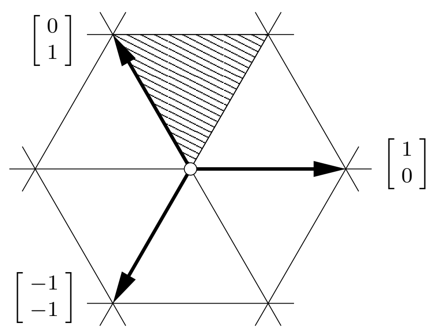

Fig. 1. A regular triangular grid.

In particular, let $B_0(\mathbf{u})$ be the piecewise linear box spline over this triangular grid defined by

$$B_0(\mathbf{i}) = \begin{cases} 1, & \text{if } \mathbf{i} = \mathbf{0} \\ 0, & \text{if } \mathbf{i} \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \end{cases}$$

and let

$$B_m(\mathbf{u}) = B_0(\mathbf{u}) * \overset{m}{\ldots} * B_0(\mathbf{u})$$

be the m-fold convolution of it. A linear combination of these basis functions $B_m(\mathbf{u})$

$$\mathbf{s}(\mathbf{u}) = \sum_{\mathbf{i} \in \mathbb{Z}^2} \mathbf{c_i} B_m(\mathbf{u} - \mathbf{i})$$

forms a box spline surface of order m. The control net of $\mathbf{s}(\mathbf{u})$ is a regular triangular net with the vertices $\mathbf{c_i}$. Any triangle of this net with the next m rings of surrounding triangles is called a B-primitive of order m, see Figure 2. Every B-primitive determines one triangular polynomial patch of $\mathbf{s}(\mathbf{u})$, see [Prautzsch and Boehm 2002].

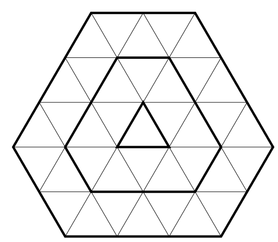

Fig. 2. A B-primitive of order 2, schematically.

By definition, a box spline surface has a regular control net. With the symmetric box splines $B_m(\mathbf{u})$ it is possible to extend this definition. Here, in this paper a **box spline surface of order** m with an arbitrary, i.e., not necessarily regular triangular net, consists of the patches defined by all B-primitives of order m contained in the net. For simplicity, we only consider nets without boundary. Then all vertices with valence $n \neq 6$ are **irregular**. If all irregular vertices are surrounded by at least $2m$ rings of regular vertices, each of them corresponds bijectively to an n-sided hole in the box spline surface. Figure 3 shows a box spline surface of order 2 and its control net. Note that it is impossible to define box spline surfaces with b-primitives that

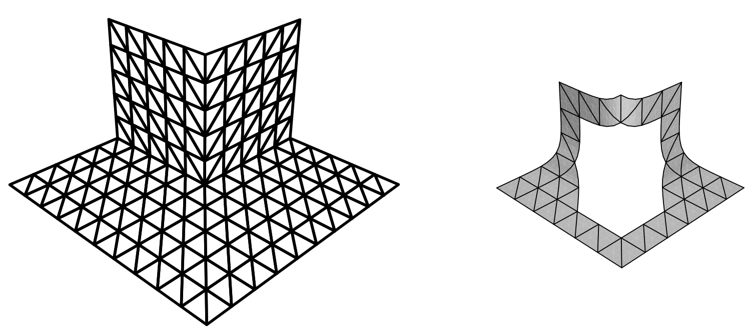

Fig. 3. A box spline surface of order 2 (right) and its control net (left).

are not symmetric if the control net has irregular vertices.

The "size" of an n-sided hole in a box spline surface $\mathbf{s}(\mathbf{u})$ depends on the order m. The hole boundary is formed by $n \cdot m$ patches of $\mathbf{s}(\mathbf{u})$, where we do not count patches with only one corner on the hole boundary. We wish to fill these holes smoothly and first show how to reduce their size. The k-ring of a vertex $\mathbf{c}$ consists of all vertices that are not further away from $\mathbf{c}$ than k edges, i.e. it consists of $\mathbf{c}$ and the next k rings of vertices around it. In particular, the 0-ring of $\mathbf{c}$ sonsits only of $\mathbf{c}$ and a (-1)-ring is empty. Let $k \leq m$ and let the k-ring of any irregular vertex coalesce into one multiple vertex as shown in Figure 4.

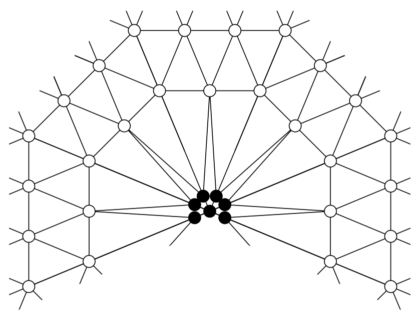

Fig. 4. An irregular vertex as a multiple vertex for $k = 1$.

Further, we treat irregular vertices specially. To explain how, it suffices to consider a net with one irregular vertex of valence $n \neq 6$. This net consists of n regular net segments $C_1, \ldots, C_n$ that are topologically equivalent to a regularly subdivided cone, as shown by heavy lines in Figure 5 left. We count periodically, i.e., $C_i = C_{i+n}$, and assume that C_i is adjacent to C_{i+1}.

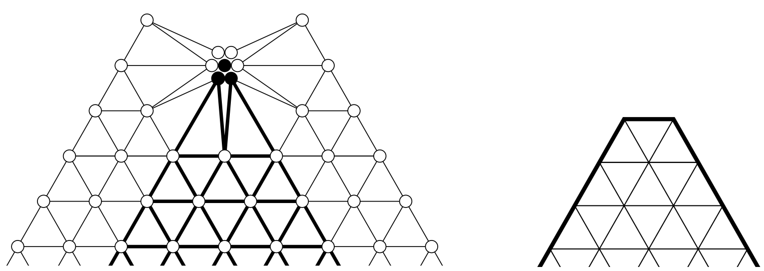

Fig. 5. A net segment D_i (left) and the associated box spline surface $\mathbf{d}_i$ for order $m = 2$ and $k = 1$ (right), schematically.

To get to the surface, we momentarily remove the $(m-k-1)$-ring of the irregular vertex. What remains of one segment C_i is equivalent to an obtuse cone. Then, we add to the remains of C_i the next m layers of control points (and thus also re-insert the points momentarily removed). Finally, we replace the irregular k-ring by a regular degenerate k-ring at the same position such that we obtain a regular net D_i as shown schematically in Figure 5 left. The net D_i defines a box spline surface $\mathbf{d}_i(\mathbf{u})$ of order m as shown in Figure 5 right.

Since D_i and D_{i+1} are part of a regular net, $\mathbf{d}_i(\mathbf{u})$ and $\mathbf{d}_{i+1}(\mathbf{u})$ have a C^{2m} joint. Consequently, $\mathbf{d}_1(\mathbf{u}), \ldots, \mathbf{d}_n(\mathbf{u})$ form a C^{2m} surface with a hole whose boundary is formed by $n(m-k)$ patches. We call it a *b*-surface of type mk, short a b_{mk}-surface.

If $n < 6$, we obtain a k_{mk}-surface in exactly the same fashion. However, the net segments D_i have coalescing vertices in this case and are not just simple subsets of the entire net. Therefore, it is easier to see what happens if we double the net and view it as a two-sheeted net winding around the irregular vertex twice. Thus, the valence of the irregular vertex is doubled and as above, we obtain a b_{mk}-surface. The surface has two sheets winding around its hole twice. Removing the extra sheet, we finally obtain a b_{mk}-surface with an n-sided hole.

Note that a b_{mm}-surface has no holes. Its derivatives up to order $2m$ are zero at extraordinary points, but, in general, it is not a C^{2m}-manifold. Since a b_{mm}-surface is subdividable, we can conclude from [Prautzsch and Reif 1999] that vicinities of extraordinary points have regular C^{2m} parametrizations only if they are planar. Also note that a generic b_{mk}-surface with $k < m$ has no singularities unless the control net further degenerates.

In the following, we assume that we have a given b_{mk}-surface, where $k = m-1$ or $k = m-2$. It is our goal to show that the holes of this surface can be filled smoothly for arbitrary high orders m.

3. FILLING HOLES IN BOX SPLINE SURFACES

Let $r \geq 2$ be an arbitrary integer and consider an n-sided hole of a b_{mk}-surface, where $k = m-1$ or $k = m-2$. The hole can be filled smoothly with $4n$ triangular patches of degree $\max\{(3m+1)r, 8m+1\}$ obtained from a best filling polynomial of degree r that we reparametrize, split and modify. The construction is based on the ideas introduced in [Prautzsch 1997; Prautzsch and Umlauf 2000].

To describe it, we need what we call a pre-C^p joint. Let $\mathbf{a}_{ijk}$ and $\mathbf{b}_{ijk}$ be the Bézier points of two triangular patches $\mathbf{a}$ and $\mathbf{b}$, respectively. Let $\mathbf{a}_{0jk} = \mathbf{b}_{0jk}$. The patches $\mathbf{a}$ and $\mathbf{b}$ have a pre-C^p joint along the corresponding boundary curve if $\mathbf{a}_{ijk}$ and $\mathbf{b}_{ijk}$ for $i \leq p$ and ($j \leq 2p$ or $k \leq 2p$) are as if $\mathbf{a}$ and $\mathbf{b}$ had a C^p joint.

CONSTRUCTION 3.1.

First, we construct a planar, piecewise polynomial n-sided macro patch $\mathbf{x}(\mathbf{u})$ of degree $3m+1$ consisting of $4n$ triangular patches $\mathbf{x}_i(\mathbf{u})$ with C^{2m} joints except between the inner patches $\mathbf{x}_1, \ldots, \mathbf{x}_n$, see Figure 6. For symmetry reasons, we construct $\mathbf{x}_i, \mathbf{x}_{i+n}, \mathbf{x}_{i+2n}$ and $\mathbf{x}_{i+3n}$ to be a rotation of $\mathbf{x}_1, \mathbf{x}_{1+n}, \mathbf{x}_{1+2n}$ and $\mathbf{x}_{1+3n}$ by $\frac{i}{n}2\pi$, respectively. The details are given in Section 4.

Second, let $\mathbf{q}(\mathbf{u})$ be polynomial of degree r and let

$$\mathbf{p}_i(\mathbf{u}) = \mathbf{q}(\mathbf{x}_i(\mathbf{u})), \qquad i = 1, \ldots, 4n.$$

The C^{2m} joints of the patches $\mathbf{x}_i(\mathbf{u})$ are carried over to the patches $\mathbf{p}_i(\mathbf{u})$. Note that $\mathbf{p}_i$ is of degree $(3m+1)r$.

If $\mathbf{q}(\mathbf{x})$ is determined appropriately, then the n-sided surface $\mathbf{p}$ formed by the $\mathbf{p}_i$ lies "in" the n-sided hole of the b_{mk}-surface, but we need to modify the boundary of $\mathbf{p}$ to obtain a C^{2m} joint with the b_{mk}-surface. Locally, the b_{mk}-surface is a box

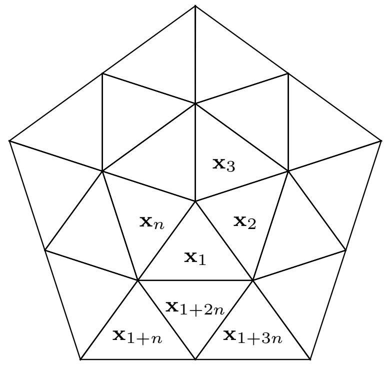

Fig. 6. The patches $\mathbf{x}_i$ schematically.

spline surface which can be extended into the hole by further patches. This means that we can change any patch $\mathbf{p}_i, i \geq n+1$, such that it has a C^{2m} joint with the b_{mk}-surface and even such that it has a pre-C^{2m} joint with any adjacent patch $\mathbf{p}_j$. The Bézier points not involved in the pre-C^{2m} joints can be changed so as to obtain full C^{2m} joints, see e.g. [Prautzsch et al. 2002].

Thus, for the modified patches $\mathbf{p}_i(\mathbf{u})$, some Bézier points depend on $\mathbf{q}(\mathbf{x})$, some depend on the b_{mk}-surface and some are constrained by the C^{2m} joints between the $\mathbf{p}_i(\mathbf{u})$. All other Bézier points can be chosen arbitrarily. These points together with $\mathbf{q}(\mathbf{x})$ can be determined such that the $\mathbf{p}_i(\mathbf{u})$ minimize a fairness functional like thin plate energy or other functionals involving higher order derivatives.

Next, we present two different parametrizations $\mathbf{x}(\mathbf{u})$.

4. A SINGULAR PARAMETRIZATION FOR THE FILLING

First, we present a singular parametrization $\mathbf{x}^b = \mathbf{x}$ of degree $3m+1$. It has similar properties as the parametrization used in [Reif 1998]. The map $\mathbf{x}^b(\mathbf{u})$ is a b_{mm}-surface controlled by an $(m+2)$-ring. The $n(m+2)$ boundary control points lie equally spaced on a regular planar n-gon and similarly the next ring of control points while all other points lie at the center as illustrated in Figure 7. The boundary triangles are isosceles triangles with vertex angle $2\pi/n$.

Theorem 4.1. *The map $\mathbf{x}^b(\mathbf{u})$ is injective and, except for its center point, regular.*

Proof. It suffices to consider the patch $\mathbf{x}_1(\mathbf{u})$. We choose the multiple control point as the origin and the symmetry axis of $\mathbf{x}_1(\mathbf{u})$ as the v-axis of the coordinate system as shown in Figure 7. We assume that the triangle $[0\,0]^t, [1\,1]^t, [0\,1]^t$ is the parameter domain of $\mathbf{x}_1(\mathbf{u})$, see Figure 1. The partial derivatives $\frac{\partial}{\partial u}\mathbf{x}_1$ and $\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\mathbf{x}_1$ are controlled by certain edge directions of the control net. We call these the u- and uv-directions.

Any real interval I of angles defines a pointed cone

$$I_c = \left\{ r \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \middle| r \geq 0, \varphi \in I \right\}.$$

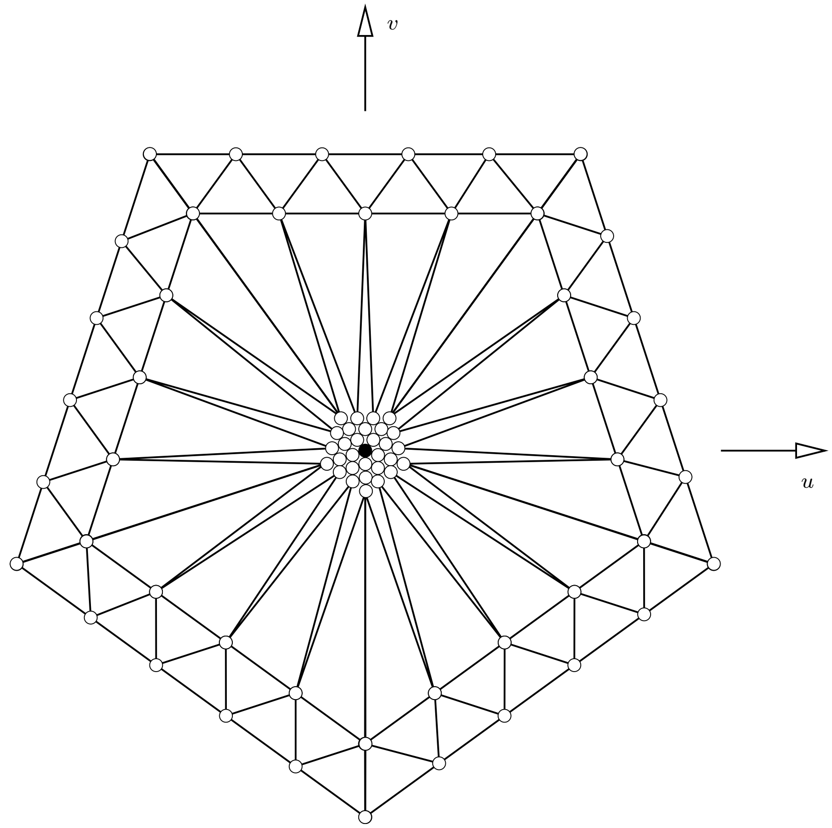

Fig. 7. The control net of a singular parametrization $\mathbf{x}^b$ for $n = 5$ and $m = 3$.

The u-direction of all u-edges in the left half-plane $u \le 0$ lie in the cone

$$A = \begin{cases} (\frac{\pi}{n} - \frac{\pi}{2}, 0]_c & , \text{ if } n \ge 6 \\ (\frac{\pi}{n} - \frac{\pi}{2}, \frac{3\pi}{n} - \frac{\pi}{2})_c & , \text{ if } n \le 5 \end{cases} \tag{1}$$

and the uv-directions of all uv-edges starting in the half-plane $u \le 0$ lie in the cone

$$B = \begin{cases} [\frac{2\pi}{n}, \frac{\pi}{2} + \frac{\pi}{n})_c & , \text{ if } n \ge 4 \\ [\frac{\pi}{2}, \frac{5}{6}\pi)_c & , \text{ if } n = 3. \end{cases}$$

Subdividing the control net (with the scaling factor 2) means that edge directions are halved and averaged by Boehm's mask shown in Figure 8. These operations preserve the cone properties above if all directions averaged lie in A or B. Because of symmetry, the u-edges crossing the v-axis are perpendicular to this axis before and after subdivision. This implies that the directions of uv-edges starting at the v-axis lie in B (and the half-plane $u \le 0$) before and after subdivision. Thus, in summary, subdivision preserves the cone properties which implies that the derivatives $\frac{\partial}{\partial u}\mathbf{x}_1$ and $\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\mathbf{x}_1$ lie in A and B, respectively. Because of the non-zero control edges, the derivatives are non-zero except at $\mathbf{0}$. Therefore, we can argue as in

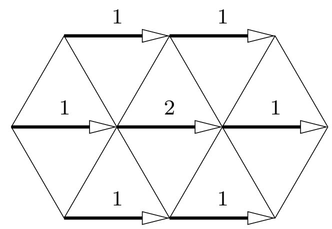

Fig. 8. Boehm's mask.

[Umlauf 2004] and show that the "left" part of $\mathbf{x}_1(u,v)$, where $v \geq u$ and $(u,v) \neq \mathbf{0}$, is injective and regular. Due to symmetry $\mathbf{x}_1(\mathbf{u})$ is injective and regular for all $\mathbf{u} \neq \mathbf{0}$.

Further, $\mathbf{x}_1(u,v) = \mathbf{x}_1(\mathbf{0})$ implies $\mathbf{x}_1(u,v) = \mathbf{x}_1(v,u)$ since $\mathbf{x}_1(\mathbf{0}) = \mathbf{0}$ lies on the symmetry axis of $\mathbf{x}_1$. Because of injectivity, we obtain $u = v$. Since the half open line segment $(\mathbf{0}, (1/2, 1/2)]$ is mapped injectively into the v-axis, the continuous map $\mathbf{x}_1$ is also injective on the closure. Hence, $\mathbf{0}$ is the only point mapped onto $\mathbf{0}$ under $\mathbf{x}_1$, which concludes the proof. □

5. A REGULAR PARAMETRIZATION FOR THE FILLING

Second, we present a regular parametrization $\mathbf{x}$ of degree $3m+1$. It is a modification of the singular parametrization in Section 4.

Let $\mathbf{x}$ be as in Section 4 and let $\mathbf{x}_{ijk}, i+j+k = 3m+1 = d$, be the Bézier points of $\mathbf{x}_1(\mathbf{u})$ such that $\mathbf{x}_{d00} = \mathbf{x}_1(\mathbf{0}) = \mathbf{0}$, $\mathbf{x}_{0d0} = \mathbf{x}_1(0,1)$ and $\mathbf{x}_{00d} = \mathbf{x}_1(1,1)$ are the corner points. We change the points $\mathbf{x}_{ijk}$ for $i < d$ by

$$\mathbf{x}_{ijk} := \frac{\varepsilon}{d}(j\,\mathbf{x}_{0d0} + k\,\mathbf{x}_{00d})$$

where $\varepsilon > 0$.

LEMMA 5.1. *The map $\mathbf{x}_1$ is regular and injective for sufficiently small ε.*

PROOF. The curve $\mathbf{a}(v) = \mathbf{x}_1(\alpha v, v), \alpha \in [0, 1/2]$ has the Bézier points

$$\mathbf{a}_i = \begin{cases} (1-\alpha)\mathbf{x}_{d-i,i,0} + \alpha\mathbf{x}_{d-i,0,i} & , i < d \\ \mathbf{x}_1(\alpha, 1) & , i = d. \end{cases}$$

Hence,

$$\Delta\mathbf{a}_i \in \left(\frac{\pi}{2} - \delta, \frac{\pi}{2} + \frac{\pi}{n} + \delta\right)_c$$

with δ depending on ε and $\lim\limits_{\varepsilon \to 0} \delta = 0$. The cross derivative curve $\frac{\partial}{\partial u} x_1(\alpha v, v)$ has the Bézier points

$$\mathbf{b}_i = \begin{cases} \varepsilon(\mathbf{x}_{00d} - \mathbf{x}_{0d0}) & , i < d-1 \\ \frac{\partial}{\partial u}\mathbf{x}_1(\alpha, 1) & , i = d-1. \end{cases}$$

For $\alpha \leq 1/2$, these points lie in the cone A, see (1). It is straight forward to conclude from these estimates that $\mathbf{x}_1$ is regular and injective for sufficiently small ε. □

Changing also $\mathbf{x}_2, \ldots, \mathbf{x}_n$ in a similar fashion we obtain a regular and injective map $\mathbf{x}$.

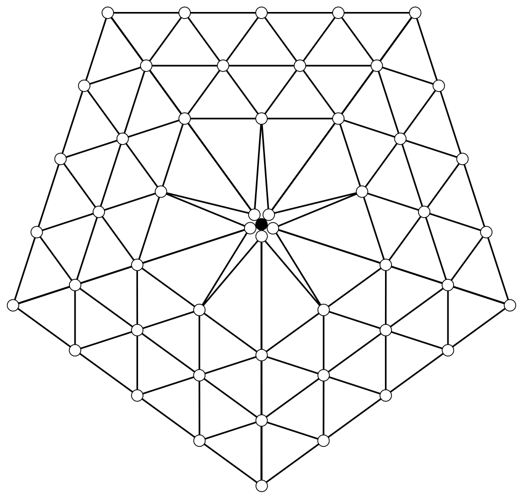

Fig. 9. The control net of the outer patches $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$ for $m = 2$.

REMARK 5.2. *In the construction above, the outer patches* $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$ *are determined by a control net as shown in Figure 7. Instead, we could also use a control net as shown in Figure 9. Then, it seems possible to construct* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *as regular injective maps with* C^{2m} *contact to the outer patches* $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$. *For* $m = 1$ *this is done in [Prautzsch and Umlauf 2000]. For* $m = 2$ *and* $n = 5$ *the Bézier points are shown in Figure 10.*

6. HALF-BOX SPLINE SURFACES

A b_{mk}-surface has even smoothness order. To obtain similar surfaces with odd smoothness orders, we are using symmetric half-box splines. In this section, we recall the definition of half-box splines and in the next section, we show how to fill a hole in a half-box spline surface in analogy to the construction for b_{mk}-surfaces.

The piecewise constant half-box splines over the triangular grid shown in Figure 1 are translates of the two functions

$$H_0^{\triangle}(\mathbf{u}) = \begin{cases} 1, & \text{if } \mathbf{u} \in \triangle \\ 0, & \text{else} \end{cases} \qquad \text{and} \qquad H_0^{\bigtriangledown}(\mathbf{u}) = \begin{cases} 1, & \text{if } \mathbf{u} \in \bigtriangledown \\ 0, & \text{else} \end{cases},$$

where $\triangle$ and $\bigtriangledown$ are the two triangles

$$\triangle := \{\mathbf{u} \,|\, 0 \le u \le v < 1\} \qquad \text{and} \qquad \bigtriangledown := \{\mathbf{u} \,|\, 0 \le v < u < 1\}$$

which form a partition of the unit square. A convolution with the box spline $B_m(\mathbf{u})$ gives the symmetric half-box splines

$$H_m^{\triangle}(\mathbf{u}) = H_0^{\triangle}(\mathbf{u}) * B_m(\mathbf{u}) \qquad \text{and} \qquad H_m^{\bigtriangledown}(\mathbf{u}) = H_0^{\bigtriangledown}(\mathbf{u}) * B_m(\mathbf{u})$$

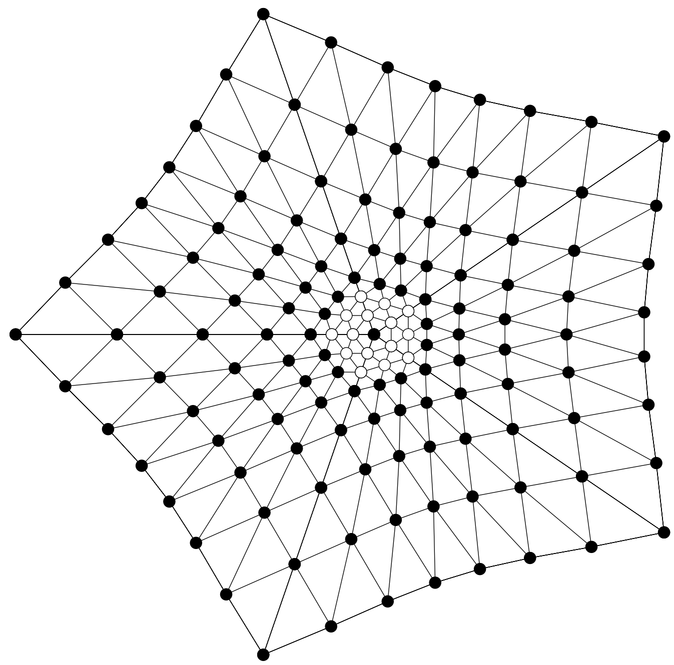

Fig. 10. The Bézier points of the inner patches $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for $m = 2$.

of order m. They are C^{2m-1} continuous and polynomial of degree $3m$ on each triangle of the grid. We use them to build a half-box spline surfaces

$$\mathbf{s}(\mathbf{u}) = \sum_{\mathbf{i} \in \mathbb{Z}^2} \left(\mathbf{c}_\mathbf{i}^{\triangle} H_m^{\triangle}(\mathbf{u} - \mathbf{i}) + \mathbf{c}_\mathbf{i}^{\bigtriangledown} H_m^{\bigtriangledown}(\mathbf{u} - \mathbf{i}) \right)$$

of order m. The control net of $\mathbf{s}(\mathbf{u})$ is a regular hexagonal net with the vertices $\mathbf{c}_\mathbf{i}^{\triangle}$ and $\mathbf{c}_\mathbf{i}^{\bigtriangledown}$. Any vertex of this net with the next m rings of surrounding hexagons is called an H-primitive of order m, see Figure 11. Every H-primitive determines one triangular polynomial patch of $\mathbf{s}(\mathbf{u})$, see [Prautzsch and Boehm 2002].

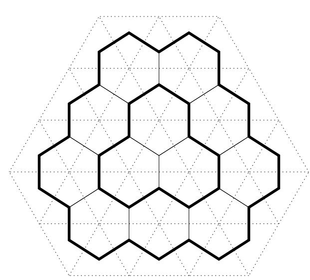

Fig. 11. An H-primitive of order 2, schematically. It is dual to the dotted B-primitive of order 2.

Half-box spline surfaces can be generalized in analogy to box spline surfaces. For this we are using a duality. A net $\mathcal{N}$ and a triangular net $\mathcal{T}$ are called **dual** if there is a one-one correspondence between the vertices of $\mathcal{N}$ and the faces of $\mathcal{T}$ such that vertices with a common edge correspond to faces with a common edge and vice versa. If two triangles in $\mathcal{T}$ coincide, their dual vertices in $\mathcal{N}$ also coincide. Under this definition, H-primitives of order m are dual to B-primitives of order m. In particular, if the vertices of $\mathcal{N}$ are the centroids of their dual triangles in $\mathcal{T}$, we call $\mathcal{N}$ the **centroid net** of $\mathcal{T}$.

A general half-box spline surface of order mk, short an h_{mk}-surface, has a control net $\mathcal{N}$ that is dual to the control net of a b_{mk}-surface. It consists of all patches determined by the H-primitives in $\mathcal{N}$ that are dual to the B-primitives in $\mathcal{T}$. Thus an h_{mk}-surface is piecewise polynomial of degree $3m$ and $2m-1$ times continuously differentiable.

As in Section 2, an h_{mm}-surface has no holes. Its derivatives up to order $2m-1$ are zero at extraordinary points, but, in general, it is not a C^{2m-1}-manifold.

7. FILLING HOLES IN HALF-BOX SPLINE SURFACES

Let r be an arbitrary integer and consider an n-sided hole of an $h_{m,k}$-surface, where $k = m-1$ or $k = m-2$. The hole can be filled smoothly with $4n$ triangular patches of degree $\max\{3mr, 8m-3\}$ in complete analogy to Section 3. Below, we show how to "dualize" the singular and the regular parametrizations $\mathbf{x}(\mathbf{u})$ given in Sections 4 and 5.

7.1 A singular parametrization for the filling

Let map $\mathbf{x}^h(\mathbf{u})$ be the h_{mm}-surface controlled by the centroid net of the control net of $\mathbf{x}^b(\mathbf{u})$ given in Section 4, see Figure 12. It is a singular parametrization to fill the hole of a h_{mk}-surface by Construction 3.1.

THEOREM 7.1. *The map $\mathbf{x}^h(\mathbf{u})$ is injective and, except for its center point, regular.*

PROOF. The control vectors of the derivative $\frac{\partial}{\partial u}\mathbf{x}^h$ form the centroid net of the control net of $\frac{\partial}{\partial u}\mathbf{x}^b$ and similarly for the derivative $\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\mathbf{x}^h$. This hexagonal centroid net can be split into two triangular nets, see [Prautzsch and Boehm 2002]. Subdividing the hexagonal net means to duplicate its vertices and to average each triangular net using Boehm's mask shown in Figure 8. Therefore, we can continue exactly as in the proof of Theorem 4.1. This will prove this theorem also. $\square$

7.2 A regular parametrization for the filling

As shown in Section 5, we can change the singular parametrization $\mathbf{x}^h$ above into a regular and injective parametrization of the same degree. Again it is possible to define the outer patches $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$ by a less degenerate control net as seen in Figure 13. The Bézier net of $\mathbf{x}_1, \ldots, x_n$ for $m = 2$ and $n = 5$ is shown in Figure 14.

8. CONCLUSION

We have introduced box and half-box spline surfaces with multiple extraordinary points to minimize the holes of general box and half-box spline surfaces with arbi-

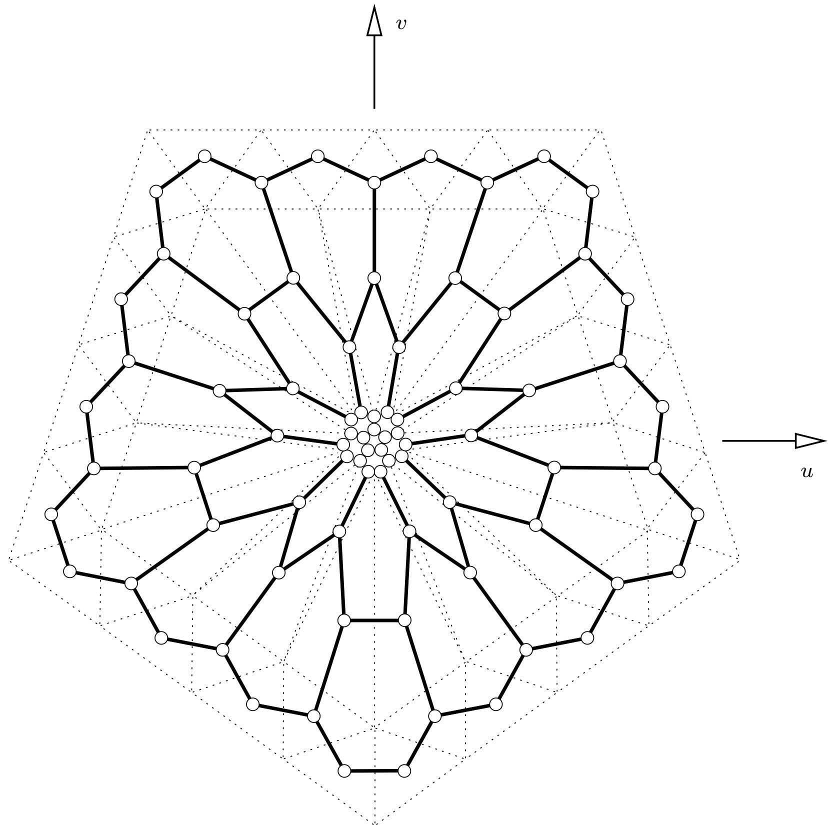

Fig. 12. The control net of a singular parametrization $\mathbf{x}$ for $n = 5$ and $m = 2$.

trary triangular control nets. Second, we have proved that the holes can be filled smoothly with a small number of polynomial patches of low degree. We have presented two solutions for box spline surfaces, as well as for half-box spline surfaces. The first solution consists of a singularly and the second of a regularly parametrized piecewise polynomial filling. It is simple to prove the correctness of the first solution, but it has the disadvantage of being singular. Therefore, the second solution should be preferred in practical applications.

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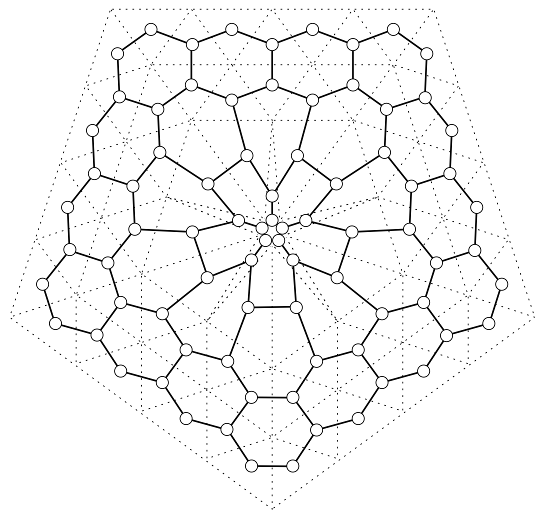

Fig. 13. The control net of the outer patches $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$ for $m = 2$.

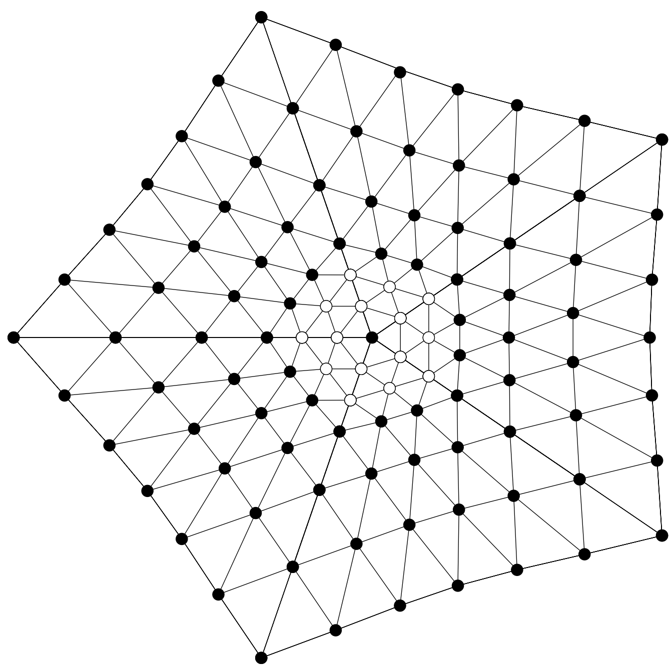

Fig. 14. The Bézier points of the inner patches $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for $m = 2$.

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