B-splines with arbitrary connection matrices

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Abstract

We consider a space of Chebyshev splines whose left and right derivatives satisfy linear constraints that are given by arbitrary non-singular connection matrices. We show that for almost all knot sequences such spline spaces have basis functions whose support is equal to the support of the ordinary B-splines with the same knots. Consequently, there are knot insertion and evaluation algorithms analogous to de Boor's algorithm for ordinary splines.

Keywords

Geometric continuity, connection matrices, knot insertion, Chebyshev splines, B-splines, rational splines.

AMS classification

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1 Introduction

A strictly increasing sequence $\alpha_0, \ldots, \alpha_k$ of knots with multiplicities $\mu_0, \ldots, \mu_k$ defines a non-decreasing knot sequence $a_0, \ldots, a_\mu$, where $\mu = \mu_0 + \ldots + \mu_k - 1$ and $a_i = \alpha_j$ if $0+\mu_0+\ldots+\mu_{j-1} \leq i < \mu_0+\ldots+\mu_j$. We assume $\mu_0 = \mu_k = n+1$ and $\mu_i \leq n$ for $i = 1, \ldots, k-1$. To each knot α_i, we associate a non-singular square *connection matrix* C_i of dimension $n+1-\mu_i$.

In 1988 Dyn and Micchelli studied piecewise polynomial splines $s(x)$ such that

$$s(x) = 0 \text{ for } x \notin [\alpha_0, \alpha_k],$$

$$(1.1) \qquad s|_{[\alpha_i, \alpha_{i+1})} \text{ is a polynomial of degree } \leq n \text{ for } i = 0, \ldots, k-1$$

$$\text{and} \quad \mathbf{d}_i^- C_i = \mathbf{d}_i^+ \text{ for } i = 1, \ldots, k-1,$$

where $\mathbf{d}_i^-$ and $\mathbf{d}_i^+$ denote the left and right $(n-\mu_i)$-jets $[s, s', \ldots, s^{(n-\mu_i)}]$ of s at $x = \alpha_i$, respectively.

One of their major result is the following [DM88, Thm 4.6]. If each connection matrix C_i is non-singular, upper triangular, totally positive and has $[1, 0, \ldots, 0]$ as first row, then the space spanned by the splines (1.1) has a basis of functions $B_0, \ldots, B_m, m = \mu - \mu_k$, with

$$(1.2) \qquad \operatorname{supp} B_i = [a_i, a_{i+n+1}] \ ,$$

$$(1.3) \qquad \sum_{i=0}^{m} B_i(x) = 1 \quad \text{for} \quad x \in [\alpha_0, \alpha_k) \ .$$

Based on this result, Seidel generalized a result for ordinary B-splines due to [Ram87] and showed that the control points $[1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1]$ of the normal curve $[B_0, \ldots, B_m]$ in $I\!R^{m+1}$ are the intersections of certain osculating flats at successive knots, see [Sei92]. Further, he concluded from this that there is a knot insertion algorithm for the splines (1.1) similar to the knot insertion algorithm for ordinary splines [Boe80], but with piecewise rational weights.

The results mentioned above were later generalized to Chebyshev splines. In [Pot93, MP96], the connection matrices C_i are identity matrices. In [Maz99, Bar96], they are upper triangular and totally positive.

It should be noted that the basis functions with totally positive connections matrices are variation diminishing [Mic95, Thm. 3.12; Bar96]. However, the total positivity of the connection matrices is not necessary for the variation diminishing property. For example, the γ-B-splines [Boe85] are variation diminishing and non-negative if and only if $\gamma \geq 0$, while the triangular connection matrices are totally positive if and only if $\gamma \in [0, 1]$.

So, even non-totally positive connection matrices can lead to well-behaved B-splines and if they do not, it is still an intriguing question whether there are basis functions satisfying (1.2) and (1.3). Currently, the only result in that direction is [DEM87], where the authors prove the existence of basis functions satisfying (1.2) and (1.3) for uniformly spaced knots and identical non-singular connection matrices $C_0 = \ldots = C_k$.

It is the purpose of this paper to show that, for Chebyshev splines, basis functions with (1.2) and (1.3) exist for almost arbitrary knot positions and arbitrary non-singular connection matrices. More precisely, given the connection matrices C_i, the feasible knot sequences $\alpha_0, \ldots, \alpha_k$ leading to "B-splines" form an open dense subset of the set of all possible knot sequences.

In Section 6, we give some further results on the derivatives and weight points, which are needed besides the control points to specify a spline in a projective space.

2 Preliminaries

An $n + 1$-dimensional space of C^n functions over some real interval I is an *extended Chebyshev space* (EC-space) if and only if all its non-zero elements have at most n zeroes, where multiplicities are counted [Sch81, Thm 2.33].

Analytically, this means that any C^n functions $u_0, \ldots, u_n$ span an $(n+1)$-dimensional EC-space $\mathcal{U}$ if and only if the determinants

$$D \begin{pmatrix} t_0 \ldots t_n \\ u_0 \ldots u_n \end{pmatrix} := \det[u_i^{(r_j)}(t_j)]_{i,j=0}^n \tag{2.1}$$

are non-zero for all non-decreasing sequences $t_0, \ldots, t_n$ in I, where the multiplicities

$$r_j := \max\{l | t_{j-l} = \ldots = t_j\}$$

determine derivatives of u_i.

Furthermore, if the determinants

$$D \begin{pmatrix} t_0 \ldots t_\nu \\ u_0 \ldots u_\nu \end{pmatrix}, \quad \nu = 0, \ldots, n,$$

are non-zero for all non-decreasing sequences $t_0, \ldots, t_\nu$, then $\mathcal{U}$ is called an *extended complete Chebyshev space* (ECC-space). It is well-known that an

EC-space over a closed, bounded interval is an ECC-space, e.g., see [Pot93, Thm 1.3].

The geometric interpretation of this is that any C^n functions $u_0, \ldots, u_n$ span an $(n+1)$-dimensional EC-space if and only if the so-called *normal curve* represented by the homogenous coordinate vector $\mathbf{u} = [u_0, \ldots, u_n]$ is of *geometric order* n [Pot93, Lemma 1.2], which means that its image $\mathbf{u}(I) = \{\mathbf{u}(t) | t \in I\}$ spans an n-dimensional projective space $\mathcal{P}^n$ and is intersected by any hyperplane in at most n points, where multiplicities are counted.

If there is a hyperplane that does not intersect $\mathbf{u}(I)$ at all, then $\mathbf{u}(I)$ lies in an affine subspace of $\mathcal{P}^n$. In particular, if $\operatorname{span}\{u_0, \ldots, u_n\}$ contains constants (without loss of generality we may assume $u_0 + \ldots + u_n = 1$), then $\mathbf{u}(I)$ does not intersect the hyperplane $x_0 + \ldots + x_n = 0$ and its affine hull is represented by the affine hyperplane $x_0 + \ldots + x_n = 1$ of $\mathbf{R}^{n+1}$. Hence, the reader wary of projective spaces may assume $\sum u_i = 1$ and replace the term "projective" by "affine" throughout most of the paper except for the final section.

The results of this paper are crucially based on a theorem by Scherk [Scherk37], which is also quoted in [Pot93] and which we state after the following definition.

Definition 2.2 *The **osculating flats** of a C^n curve $\mathbf{u}$ are denoted by*

$$O_t^i \mathbf{u} := \text{projective hull}\{\mathbf{u}(t), \mathbf{u}'(t), \ldots, \mathbf{u}^{(i)}(t)\}, \quad i = 0, \ldots, n.$$

For later reference, we point out that the osculating flats $O_t^i \mathbf{u}$ of a C^n curve $\mathbf{u}$ in $\mathcal{P}^n$ of geometric order n are non-degenerate for all t and $i = 0, \ldots, n$. Namely, if the dimension of $O_t^i \mathbf{u}$ was lower than i, then $\mathbf{u}(t), \ldots, \mathbf{u}^{(n)}(t)$ would lie in a hyperplane that has an $n+1$ fold intersection with $\mathbf{u}(I)$ at t.

Theorem 2.3 [Scherk37] *A C^n curve $\mathbf{u}$ of geometric order n has at most n osculating hyperplanes $O_t^{n-1}\mathbf{u}$ containing a given point $\mathbf{p}$.*

Consequently, if $\mathbf{u}$ is a C^n curve in $\mathcal{P}^n$of geometric order n, then any n of its osculating hyperplanes are independent. This means that any $k \leq n$ osculating hyperplanes intersect in a subspace of dimension $n-k$. Thus, we get

Corollary 2.4 *A C^n curve $\mathbf{u}$ of geometric order n has at most k osculating hyperplanes $O_t^{n-1}\mathbf{u}$ containing a given subspace V of dimension $n-k$. Hence, except for finitely many t, we get*

$$\dim(V \cap O_t^{n-1}\mathbf{u}) = \dim V - 1. \tag{2.5}$$

Note that (2.5) holds also in an affine space since there are at most $k+1$ osculating hyperplanes whose projective extensions contain the projective extension or ideal hyperplane of V. In other words, there are at most $k+1$ osculating hyperplanes parallel to V. We generalize Corollary 2.4 in the next section.

3 Intersections with osculating flats

Theorem 3.1 *Let $\mathbf{u}$ be a curve of geometric order n over an interval I and let V be any subspace of $\mathcal{P}^n$. Then,*

$$dim(V \cap O_t^{n-r}\mathbf{u}) = max\{-1, dimV - r\}$$

for $r = 0, \dots, n$ and $t \in J$, where J is a certain open dense subset with countable complement of I.

The proof of Theorem 3.1 is based on the following lemma, proposition and corollary.

Lemma 3.2 *The image of a C^n curve $\mathbf{u}$ in $\mathcal{P}^n$ over a compact interval I with non-degenerate osculating flats intersects any hyperplane $\mathcal{H}$ in at most finitely many points.*

Proof. Let $\mathbf{a}^t\mathbf{x} = 0$ be the equation of $\mathcal{H}$. Since the derivatives of $\mathbf{u}(t)$ span $\mathcal{P}^n$, there exists for every t an $r \in \{0, 1, \dots, n\}$ such that

$$\mathbf{a}^t\mathbf{u}^{(r)}(t) \neq 0 \quad \text{and} \quad \mathbf{a}^t\mathbf{u}^{(i)}(t) = 0 \quad \text{for} \quad i < r.$$

Hence, the Taylor expansion of $\mathbf{a}^t\mathbf{u}$ around t is of the form

$$\mathbf{a}^t\mathbf{u}(t+h) = \frac{h^r}{r!}\,\mathbf{a}^t\mathbf{u}^{(r)}(t) + o(h^r),$$

which is non-zero for sufficiently small h, except for $h = 0$ if $r > 0$. Consequently, any $t \in I$ has an open neighborhood in I, where $\mathbf{u}$ intersects $\mathcal{H}$ at most once. Since I is compact, finitely many of these neighborhoods cover I, whence the lemma follows. ∎

Proposition 3.3 *Let* $\mathbf{u}$ *be a* C^n *curve in* $\mathcal{P}^n$ *with non-degenerate osculating flats over some interval* I*, and let* $\mathcal{H}$ *be any hyperplane that does not intersect its image* $\mathbf{u}(I)$*. Intersecting the tangents of* $\mathbf{u}$ *with* $\mathcal{H}$ *results in a* C^{n-1} *curve* $\mathbf{v}$ *with non-degenerate osculating flats where*

$$O_t^r\mathbf{v} = O_t^{r+1}\mathbf{u} \cap \mathcal{H} \ , \quad \textit{for} \ \ r = 0, \ldots, n-1.$$

Proof. Let $\mathbf{a}^t\mathbf{x} = 0$ be the equation of $\mathcal{H}$. Then, the curve $\mathbf{v}$ can be written as

$$\mathbf{v}(t) = \mathbf{u}'(t) - \lambda(t)\mathbf{u}(t),$$

where

$$\lambda = \frac{\mathbf{a}^t\mathbf{u}'}{\mathbf{a}^t\mathbf{u}}.$$

Note that $\mathbf{a}^t\mathbf{u} \neq 0$ since we are assuming that $\mathbf{u}(I)$ does not intersect $\mathcal{H}$. So, we can easily differentiate λ and $\mathbf{v}$ and obtain

$$\mathbf{v}^{(r)} = \mathbf{u}^{(r+1)} - \sum_{i=0}^{r} \binom{r}{i} \lambda^{(i)}\mathbf{u}^{(r-i)}$$

for $r = 0, \ldots, n-1$. Consequently, $\mathbf{v}$ is a C^{n-1} curve, where $\mathbf{u}(t), \mathbf{v}^0(t), \ldots, \mathbf{v}^{(r)}$ span $O_t^{r+1}\mathbf{u}$. Therefore, $\mathbf{v}$ has non-degenerate osculating flats and $O_t^r\mathbf{v} = O_t^{r+1}\mathbf{u} \ \cap \mathcal{H}$, which completes the proof. ∎

If the hyperplane $\mathcal{H}$ in Proposition 3.3 is arbitrary, it intersects $\mathbf{u}(J)$ in at most finitely many points for every compact subinterval J of I due to Lemma 3.2. Since any interval can be covered by countably many compact subintervals, $\mathcal{H}$ intersects the entire curve image $\mathbf{u}(I)$ in at most countably many points, which have no limit points on $\mathbf{u}(I)$. In other words, $\mathcal{H}$ partitions $\mathbf{u}(I)$ into at most countably many segments which, without the intersection points, form an open dense subset $\mathbf{u}(I)$. The tangents of each segment intersect $\mathcal{H}$ in a curve for which Proposition 3.3 aplies with an arbitrary hyperplane of $\mathcal{H}$. Consequently, by successive applications of Proposition 3.3 it follows the

Corollary 3.4 *The osculating flats* $O_t^{n-r}\mathbf{u}$ *of a* C^n *curve* $\mathbf{u}$ *with non-degenerate osculating flats over some interval* I *do not intersect an arbitrary* $(r-1)$*-dimensional subspace* $\mathcal{V}$ *for* $t \in J$*, where* J *is some open dense subset of* I *with a countable complement.*

As pointed out below Definition 2.2, the Corollary holds, in particular, for a C^n curve of geometric order n in $\mathcal{P}^n$. Hence, we can present a

Proof for Theorem 3.1. We prove the theorem by backward induction on $r = n, \ldots, 0$.

First, if $r > v := \dim V$, then the theorem follows from Corollary 3.4.

Second, if $r \leq v$, we use the dimension theorem to bound the dimension

$$d_r := \dim(V \cap O_t^{n-r}\mathbf{u})$$

from below,

$$\begin{aligned} d_r &= v + (n-r) - \dim(V \sqcup O_t^{n-r}\mathbf{u}) \\ &\geq v - r, \end{aligned}$$

and above

$$\begin{aligned} d_r &\leq v + (n-r) - \dim(V \sqcup O_t^{n-r-1}\mathbf{u}) \\ &= d_{r+1} + 1. \end{aligned}$$

Because of the induction hypothesis, this upper bound equals the lower bound, which completes the proof. ∎

4 Chebyshev splines

Given a strictly increasing sequence of knots $\alpha_0, \ldots, \alpha_k$ with multiplicities μ_i and connection matrices C_i as described in Section 1, we associate with each interval $[\alpha_i, \alpha_{i+1}]$ an ECC-space $\mathcal{U}_i$ of dimension $n+1$. The restriction of $\mathcal{U}_i$ to the half-open interval $[u_i, u_{i+1})$ is denoted by $\overline{\mathcal{U}}_i$.

The spline space $\mathcal{S}$ in which we are interested is

$$\begin{aligned} \mathcal{S} \;:=\; &\{s \mid \forall i = 0, \ldots, k-1 : s|_{[a_i, a_{i+1})} \in \overline{\mathcal{U}_i} \\ &\text{and } \forall i = 1, \ldots, k-1 : \mathbf{d}_i^+ = \mathbf{d}_i^- C_i\}, \end{aligned} \tag{4.1}$$

where $\mathbf{d}_i^-$ and $\mathbf{d}_i^+$ denote the left and right jets $[s^{(0)}, \ldots, s^{(n-\mu_i)}]$ of $s(x)$ at $x = \alpha_i$. Note that $\mathcal{S}$ contains discontinuous functions if the first column of some connection matrix C_i is not of the form $[1, 0, \ldots, 0]^t$. However, any curve $\mathbf{s}$ in some projective space whose coordinate functions are in $\mathcal{S}$ has a continuous osculating flat $O_t^{n-\mu_i}\mathbf{s}$ at $t = a_i$. We need this property also for curves in affine spaces. Therefore, if we work with affine spaces, we assume that the entry in the upper left corner of every connection matrix must be one.

For later reference, we mention the following well-known fact.

Lemma 4.2 *The dimension of the linear space $\mathcal{S}$ is $m+1 := \mu_1 + \ldots + \mu_{k-1}$ and any function u in $\mathcal{U}_i$ can be extended to a spline in $\mathcal{S}$.*

Any basis $s_0, \ldots, s_m$ of $\mathcal{S}$ forms the coordinates of a curve in $\mathcal{P}^m$. It is called a *normal curve* for $\mathcal{S}$ and it spans $\mathcal{P}^m$ since its coordinates are linearly independent. Moreover, from Lemma (4.2), it follows that the coordinates $s_0, \ldots, s_m$ restricted to any interval $[a_i, a_{i+1})$ span $\overline{\mathcal{U}}_i$. Hence, we obtain

Lemma 4.3 *A normal curve of $\mathcal{S}$ restricted to an interval $[a_i, a_{i+1})$ is of geometric order n.*

5 Control points and B-splines

In this and the next section, we come to the core part of the paper, where we construct a locally supported basis for $\mathcal{S}$. The crucial result is stated in Theorem 5.2.

First, for all $i = 0, ..., k-1$, we extend the ECC-space $\mathcal{U}_i$ to an ECC-space over a larger interval $[b_i, c_{i+1}]$, where $b_i < \alpha_i < \alpha_{i+1} < c_{i+1}$ and $c_{i+1} < b_{i+2}$. This is always possible, see [Sch81, Thm 9.3]. Then, we define

(5.1) $$K := \{(\alpha_0, \ldots, \alpha_k) | \forall i = 0, \ldots, k : b_i \leq a_i \leq c_i\}.$$

Any knot sequence $\boldsymbol{\alpha} \in K$ with the same connection matrices C_i and ECC-spaces $\mathcal{U}_i$ as before defines a spline space $\mathcal{S} = \mathcal{S}_{\boldsymbol{\alpha}}$ by (4.1).

To simplify our notation, we use the convention that a multiple intersection of an osculating flat $O_a^{n-1}\mathbf{s}$ stands for a lower dimensional flat. More precisely, let $R_i^{n-\nu}$ and $L_i^{n-\nu}$ denote the right and left osculating flats of $\mathbf{s}$ at a knot a_i, respectively. Only if a_i is a knot with multiplicity $\neq \nu$, these flats may be different. If $a_i, \ldots, a_j$ is the part

$$\underbrace{\alpha_q, \ldots, \alpha_q}_{\nu}, \underbrace{\alpha_{q+1}, \ldots, \alpha_{q+1}}_{\mu_{q+1}}, \ldots, \underbrace{\alpha_{r-1}, \ldots, \alpha_{r-1}}_{\mu_{r-1}}, \underbrace{\alpha_r, \ldots, \alpha_r}_{\lambda}$$

of the knot sequence $a_0, \dots, a_\mu$, then we define

$$O_i \cap \dots \cap O_j$$

$$:= \begin{cases} R_q^{n-\nu} \cap R_{q+1}^{n-\mu_{q+1}} \cap \dots \cap R_{r-1}^{n-\mu_{r-1}} \cap L_r^{n-\lambda}, & \text{if } a_{i+1} \neq a_j \\ R_q^{n-\nu}, & \text{otherwise.} \end{cases}$$

We call a knot sequence, $\boldsymbol{\alpha} \in K$, *$q$-regular*, $q = 0, \dots, \mu$, if

$$\dim (O_i \cap \dots \cap O_j) = n - (j - i + 1)$$

for $j = 0, \dots, q$ and $j - n \leq i \leq j$. If $\boldsymbol{\alpha}$ is μ-regular, we call it *regular* and *singular* otherwise.

Theorem 5.2 *With respect to the standard topology on K, the regular knot sequences form an open dense subset K_μ.*

Proof. First, we show that K_μ is open. For all knot sequence $\boldsymbol{\alpha} \in K$, we construct a normal curve $\mathbf{s}_{\boldsymbol{\alpha}}$ in the space $\mathcal{S}_{\boldsymbol{\alpha}}$ whose osculating flats depend continuously on $\boldsymbol{\alpha} \in K$. To do so, we extend each ECC-space $\mathcal{U}_i$ to $[b_i, c_{i+1}]$, fix a basis and restrict it to the interval $[\alpha_i, \alpha_{i+1})$. Thus, the derivatives of the basis functions at α_i and α_{i+1} depend continuously on $\boldsymbol{\alpha}$. Then, we express each segment of $\mathbf{s}_{\boldsymbol{\alpha}}$ as a linear combination of these basis functions and require that the left derivatives of order $n - \mu_i + 1, \dots, n$ of $\mathbf{s}_{\boldsymbol{\alpha}}$ at $\alpha_i, i = 0, \dots, k-1$ are the unit points of $\mathcal{P}^m$.

These interpolation conditions together with the connection conditions determine $\mathbf{s}_{\boldsymbol{\alpha}}$, i.e., the coefficients of its basis representation. The entries of the interpolation matrix are derivatives of the basis functions at the knots α_i. Since these are continuous in $\boldsymbol{\alpha}$, so are the coefficients of $\mathbf{s}_{\boldsymbol{\alpha}}$ and hence the osculating flats of $\mathbf{s}_{\boldsymbol{\alpha}}$. If the knot sequence is regular, the osculating flats of $\mathbf{s}_{\boldsymbol{\alpha}}$ intersect in a space with smallest possible dimension. Because of continuity, this remains true also in a sufficiently small neighborhood of $\boldsymbol{\alpha}$.

Second, we show that the sets K_q consisting of the q-regular knot sequences in K are dense. Let $\boldsymbol{\alpha} \in K_{q-1}$ and $a_j < a_{j+1} = \dots = a_q$. Then, $O = O_i \cap \dots \cap O_j$, where $q - n \leq i \leq j$, has dimension $n - (j - i + 1)$. Because $O \subset O_j \subset \operatorname{span} \mathbf{s}[a_j, a_{j+1})$, it follows from Theorem 3.1 that $O \cap L_q^{n-(q-j)} = O_i \cap \dots \cap O_q$ has dimension $n - (q - i + 1)$ for all positions of the knots $\alpha_q \in J$, where J is a dense open subset of $[b_q, c_q]$. Consequently, there are

q-regular knot sequences arbitrarily close to $\boldsymbol{\alpha}$ and by induction we obtain that there are also regular knot sequences arbitrarily close to $\boldsymbol{\alpha}$. Since any knot sequence is 0-regular, the theorem follows. ■

In the sequel, we only consider regular knot sequences. Hence, there are always points $\mathbf{c}_i \in \mathcal{P}^m, i = 0, \ldots, m$ uniquely defined by

$$\{\mathbf{c}_i\} = O_{i+1}^{n-1} \cap \ldots \cap O_{i+n}^{n-1}. \tag{5.3}$$

These points are *control points* of the normal curve $\mathbf{s}$ in the following sense.

Theorem 5.4 *The image of a normal curve* $\mathbf{s}$ *of* $\mathcal{S}$ *lies in the projective hull of the points* $\mathbf{c}_i$ *defined in (5.3). More specifically, for any non-empty knot interval* $[a_i, a_{i+1})$ *we have span* $\{\mathbf{c}_{i-n}, \ldots, \mathbf{c}_i\} =$ *span* $\mathbf{s}[a_i, a_{i+1})$.

Proof. We observe that for all $j = i - n + 1, \ldots, i$

$$\emptyset = O_j \cap \ldots \cap O_{j+n} = R_j^{n-1} \cap \ldots \cap R_i^{n-1} \cap L_{i+1}^{n-1} \cap \ldots \cap L_{j+n}^{n-1}.$$

Therefore, $R_j^{n-1}, \ldots, R_i^{n-1}, L_{i+1}^{n-1}, \ldots, L_{j+n}^{n-1}$ are independent hyperplanes and

$$\mathcal{A} := R_{j+q}^{n-1} \cap \ldots \cap R_i^{n-1} \cap L_{i+1}^{n-1} \cap \ldots \cap L_{j+n-1}^{n-1}$$

and

$$\mathcal{B} := R_{j+q+1}^{n-1} \cap \ldots \cap R_i^{n-1} \cap L_{i+1}^{n-1} \cap \ldots \cap L_{j+n}^{n-1}$$

are distinct hyperplanes of

$$\mathcal{C} := R_{j+q+1}^{n-1} \cap \ldots \cap R_i^{n-1} \cap L_{i+1}^{n-1} \cap \ldots \cap L_{j-n-1}^{n+1}$$

for $j + q \leq i$. Consequently, if $\mathbf{c}_{j-1}, \ldots, \mathbf{c}_{j+q-1}$ span $\mathcal{A}$ and $\mathbf{c}_j, \ldots, \mathbf{c}_{j+q}$ span $\mathcal{B}$, then $\mathbf{c}_{j-1}, \ldots, \mathbf{c}_{j+q}$ span the join $\mathcal{A} \sqcup \mathcal{B} = \mathcal{C}$. Using this argument repeatedly, the theorem follows by induction on $q = 0, \ldots, n-1$ from (5.3). ■

6 A locally supported basis

For any two bases $\{s_0^1, \ldots, s_m^1\}$ and $\{s_0^2, \ldots, s_m^2\}$ of the spline space $\mathcal{S}$, there is a non-singular matrix T such that

$$T[s_0^1, \ldots, s_m^1]^t = [s_0^2, \ldots, s_m^2]^t.$$

This means that any two normal curves $\mathbf{s}^1$ and $\mathbf{s}^2$ are projective images of each other. Moreover, since the construction of the control points $\mathbf{c}_i$ of a normal curve $\mathbf{s}$ is projectively invariant, any projective image $T\mathbf{s}$ of $\mathbf{s}$ has the control points $T\mathbf{c}_i$.

Consequently, there is a normal curve

$$\mathbf{n} = [B_0, \ldots, B_m]^t \tag{6.1}$$

whose control points are the points

$$\mathbf{e}_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Remark 6.2 *A spline in projective space is not completely determined by its control points. Since the homogeneous coordinates of a point are only determined up to a factor, all normal curves* $[\beta_0 B_0, \ldots, \beta_m B_m]^t$ *for* $\beta_0, \ldots, \beta_m \in I\!R\backslash\{0\}$ *have the same control points* $\mathbf{e}_0, \ldots, \mathbf{e}_m$*. However, if* $\mathcal{S}$ *contains constants, then, we can restrict ourselves to bases of* $\mathcal{S}$ *that form a partition of unity as explained in Section 2. Under this restriction, all normal curves are affine images of each other and the normal curve with control points* $\mathbf{e}_0, \ldots, \mathbf{e}_m$ *is uniquely defined by the stipulation* $\sum B_i = 1$*.*

Theorem 6.3 *The basis functions* B_i *defined in (5.1) are locally supported,*

$$suppB_i = [a_i, a_{i+n+1}].$$

Proof. Since any non-empty segment $\mathbf{n}[a_j, a_{j+1})$ is of geometric order n, it has exactly $n+1$ non-zero coordinates and their support is $[a_j, a_{j+1}]$. Due to Theorem 5.4, the segment $\mathbf{n}[a_j, a_{j+1})$ lies in the projective hull of $\mathbf{e}_{j-n}, \ldots, \mathbf{e}_j$. Hence, only $B_{j-n}, \ldots, B_j$ are supported over $[a_j, a_{j+1}]$ whence the Theorem follows. ∎

The support of the basis functions B_i is minimal. Namely, we have

Corollary 6.4 *The support of a spline* $s(x) = \sum_{i=0}^m c_i B_i(x)$ *in* $\mathcal{S}$ *is a union of the basis supports,*

$$supp\ \mathbf{s} = \cup_{c_i \neq 0} supp\ B_i.$$

Proof. From Lemma 4.2, it follows that all elements in $\mathcal{U}_j$ can be extended to a spline in $\mathcal{S}$. Hence, the non-zero basis splines $B_{j-n}, \ldots, B_j$ over $[a_j, a_{j+1})$ span $\mathcal{U}_j$. Since $\dim \mathcal{U}_j = n + 1$, these basis splines are also linearly independent over $[a_j, a_{j+1})$. Hence, $[a_j, a_{j+1}] \subset \operatorname{supp} \mathbf{s}$ if and only if $c_i \neq 0$ for some $i \in \{j - n, \ldots, j\}$. Further, if $c_i \neq 0$, then $[a_j, a_{j+1}] \subset \operatorname{supp} \mathbf{s}$ for all $j = i, \ldots, i + n$, which concludes the proof. ∎

The following example demonstrates the results of this section. Let $n = 3$, let

$$a_0, \ldots, a_{12} = 3, 3, 3, 3, 4, 5, 6, 7, 8, 9, 9, 9, 9$$

be the knot sequence, let $\mathcal{U}_0 = \ldots = \mathcal{U}_6$ be the space of all cubic polynomials and let

$$C_1 = C_2 = C_4 = C_5 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$$

be the associated connection matrices. Then, we can easily obtain a basis $B_0, \ldots, B_8$ for the corresponding spline space $\mathcal{S}$ from the ordinary piecewise cubic B-splines $N_0, \ldots, N_8$ over the knots $a_0, \ldots, a_{12}$, namely

$$B_i = N_i \quad \text{for} \quad i \neq 3, 5,$$

$$B_3(x) = \begin{cases} N_3(x) & \text{if } x < 6 \\ N_5(x) & \text{if } x \geq 6 \end{cases}$$

and

$$B_5(x) = \begin{cases} N_5(x) & \text{if } x < 6 \\ N_3(x) & \text{if } x \geq 6 \end{cases}.$$

This basis is illustrated in Figure 1.

The basis spline B_5 has smaller support than possible if the knots were in a regular position, see Corollary 5.4. Hence, the knots are in a singular position. If we move the knot a_6 to a new position $6 + \varepsilon$, the knot sequence becomes regular for all $\varepsilon \in (-1, 1)$, except for $\varepsilon = -\frac{1}{2}, 0, \frac{1}{2}$. The new basis functions are again the ordinary piecewise cubic B-splines over $a_0, \ldots, a_{13}$, except for $B_j, j = 3, 4, 5$. These B-splines are depicted in Figure 2 for $\varepsilon = \frac{1}{4}$. The non-zero of the 13 Bézier ordinates $b_0^j, \ldots, b_{12}^j$ of B_j are given in Table 1.

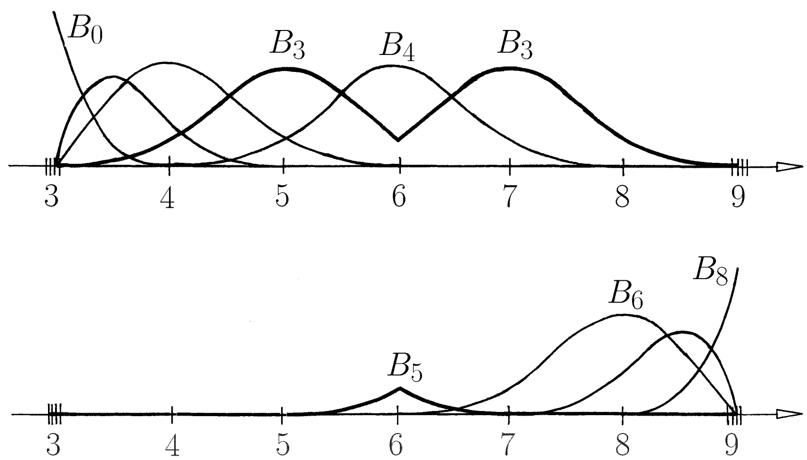

Figure 1: Basis splines for a singular knot sequence.

	b_3^j	b_4^j	b_5^j	b_6^j	b_7^j	b_8^j	b_9^j
$j=3$	$\frac{1/2}{\varepsilon+3}$	$\frac{1}{\varepsilon+3}$	$\frac{2}{\varepsilon+3}$	$\frac{6\varepsilon^2+12\varepsilon+2}{2\varepsilon^3+11\varepsilon^2+17\varepsilon+6}$	$\frac{2\varepsilon}{2\varepsilon+1}$	$\frac{\varepsilon-1}{2\varepsilon+1}$	$\frac{(\varepsilon-1)^2}{4\varepsilon^2+2\varepsilon}$
$j=4$	$\frac{1}{2\varepsilon^2+5\varepsilon+2}$	$\frac{1}{2\varepsilon+1}$	$\frac{\varepsilon+2}{2\varepsilon+1}$	$\frac{2\varepsilon^3-5\varepsilon}{4\varepsilon^3-\varepsilon}$	$\frac{\varepsilon-2}{2\varepsilon-1}$	$\frac{-1}{2\varepsilon-1}$	$\frac{1}{2\varepsilon^2-5\varepsilon+2}$
$j=5$	$\frac{(1+\varepsilon)^2}{4\varepsilon^2-2\varepsilon}$	$\frac{\varepsilon+1}{2\varepsilon-1}$	$\frac{2\varepsilon}{2\varepsilon-1}$	$\frac{-6\varepsilon^2+12\varepsilon-2}{2\varepsilon^3-11\varepsilon^2+17\varepsilon-6}$	$\frac{-2}{\varepsilon-3}$	$\frac{-1}{\varepsilon-3}$	$\frac{-1/2}{\varepsilon-3}$

Table 1: The non-zero Bèzier ordinates of B_3, B_4, B_5.

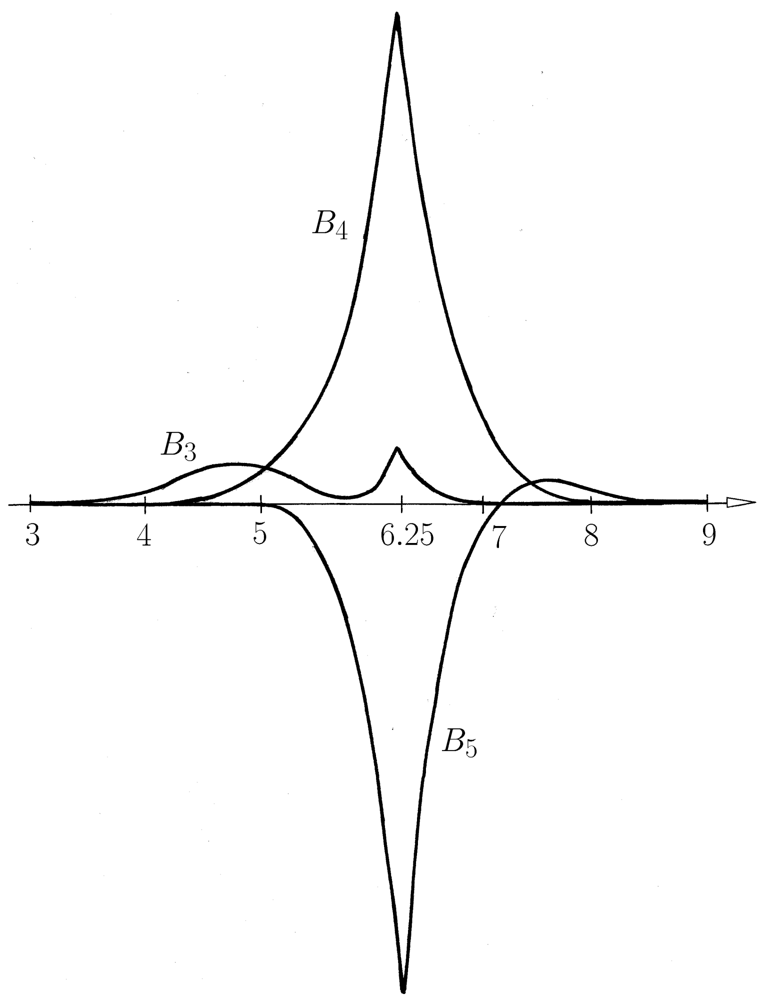

Figure 2: The basis splines for a regular knot sequence.

7 Weight points, derivatives and knot insertion

This final section contains a collection of remarks related to and relating weight points, derivatives and knot insertion.

Let $\mathbf{n} = [B_0, \ldots, B_m]^t$ be a normal curve of $\mathcal{S}$ with the unit points $\mathbf{e}_0, \ldots, \mathbf{e}_m$ as control points. Then all normal curves

$$\mathbf{m} = [\beta_0 B_0, \ldots, \beta_m B_m]^t, \tag{7.1}$$

where $\beta_i \neq 0$, have the same control points as they are projective images of $\mathbf{n}$ under the maps given by the diagonal matrices

$$T = \operatorname{diag}(\beta_0, \ldots, \beta_m).$$

Hence, the control points $\mathbf{e}_i$ alone are not enough to define any specific curve $\mathbf{m}$. Therefore, it is common to associate a weight $\beta_i \neq 0$ with each control point. The points $\mathbf{e}_i$ and the weights β_i together with the basis functions B_i define the normal curve (7.1). A more geometric way is to specify the weights β_i either by the so-called **inner weight points**

$$\overline{\mathbf{w}}_i = \beta_i \mathbf{e}_i + \beta_{i+1} \mathbf{e}_{i+1}, \qquad i = 0, \ldots, m-1 \quad ,$$

or by the **outer weight points**

$$\mathbf{w}_i = \beta_i \mathbf{e}_i - \beta_{i+1} \mathbf{e}_{i+1}, \qquad i = 0, \ldots, m-1 \quad .$$

Weight points for rational curves were introduced to CAGD by Farin [Far83] and already used in 1870 by Haase [Haa1870, BM99]. Depending on the projective coordinate system, the weight points $\mathbf{w}_i$ lie anywhere on the line $\mathbf{e}_i \mathbf{e}_{i+1}$ except on $\mathbf{e}_i$ and $\mathbf{e}_{i+1}$. Pottmann [Pot93] proposes to choose the inner weight points on certain osculating flats $O_a^{n-1}\mathbf{m}$. This has the advantage that any normal curve $\mathbf{m}$ is completely defined by its control points and weight points.

Here, we discuss a different approach which leads to a few further insights. We say that a spline in $\mathcal{S}$ is **piecewise constant** if it is constant over each knot interval $[\alpha_i, \alpha_{i+1})$. The following lemma is a direct consequence of the connection conditions at the knots α_i.

Lemma 7.2 *If $\mathcal{S}$ contains piecewise constant functions, then the first rows of the connection matrices C_i are of the form $[c_i, 0, \ldots, 0]$ and any two piecewise constant splines in $\mathcal{S}$ are multiples of each other.*

If $\mathcal{S}$ contains piecewise constant functions, Lemma 7.2 implies that up to a common factor there is only one minimally supported basis $\{B_0, \ldots, B_m\}$ of $\mathcal{S}$ such that $\sum B_i$ is piecewise constant. In other words, there is only one normal curve

$$\mathbf{n} = [B_0, \ldots, B_m]$$

of $\mathcal{S}$ with control points $\mathbf{e}_i$ and piecewise constant coordinate sum. The derivative of $\mathbf{n}$ lies in the hyperplane $x_0 + \cdots + x_m = 0$ and the derivatives of $\mathbf{m} = [\beta_0 B_0, \ldots, \beta_m B_m]$ in the hyperplane $x_0/\beta_0 + \cdots + x_m/\beta_m = 0$. This hyperplane intersects the lines $\mathbf{e}_i\mathbf{e}_{i+1}$ in the points

$$\mathbf{w}_i = \beta_i \mathbf{e}_i - \beta_{i+1}\mathbf{e}_{i+1}.$$

We define the **(outer) weight points** of any normal curve $\mathbf{s}$ of $\mathcal{S}$ to be the intersection of the control polygon of $\mathbf{s}$ with the hyperplane spanned by the derivative $\mathbf{s}'$.

Theorem 7.3 *If $\mathcal{S}$ contains piecewise constants, every normal curve $\mathbf{s}$ for $\mathcal{S}$ has different control or different weight points, i.e., the control and weight points determine a spline curve uniquely.*

Proof. As observed in the beginning of Section 6, there is a unique projective map π mapping the normal curve $\mathbf{n}$ onto any other normal curve $\mathbf{s}$. Since the construction of the control and weight points is projectively invariant, $\mathbf{s}$ has the control points $\mathbf{c}_i = \pi\mathbf{e}_i$ and the weight points $\mathbf{d}_i = \pi\mathbf{w}_i$, where $\mathbf{w}_i = \mathbf{e}_i - \mathbf{e}_{i+1}$. Since the $\mathbf{d}_i$ lie on the lines $\mathbf{c}_i\mathbf{c}_{i+1}$, there are constants $\gamma_i \neq 0$ such that $\mathbf{d}_i = \gamma_i\mathbf{c}_i - \gamma_{i+1}\mathbf{c}_{i+1}$. Hence, π is represented by the matrix $T = [\gamma_0\mathbf{c}_0 \ldots \gamma_m\mathbf{c}_m]$, which proves the theorem. ∎

Remark 7.4 *Also all lower dimensional projective images $\pi\mathbf{n}$ of the normal curve are uniquely defined by the images $\mathbf{c}_i = \pi\mathbf{e}_i$ and $\mathbf{d}_i = \pi\mathbf{w}_i$, provided that, any two successive control points $\mathbf{c}_i$ and $\mathbf{c}_{i+1}$ do not coalesce.*

If $\mathcal{S}$ contains piecewise constant functions, then $\mathbf{S}' := \{s'(x)|s(x) \in \mathcal{S}\}$ is a spline space with the knots $\alpha_0, \ldots, \alpha_k$ and the connection matrices C_i' obtained from C_i by deleting the first row and column and the ECC-spaces $\mathcal{U}_i' = \{u'(x)|u(x) \in \mathcal{U}_i\}$.

An osculating flat $O_a^\mu \mathbf{s}$ of a normal curve $\mathbf{s}$ of $\mathcal{S}$ contains the osculating flat $O_a^{\mu-1}\mathbf{s}'$ of the derivative $\mathbf{s}'$. Hence, the control points $\mathbf{c}_i'$ of $\mathbf{s}'$ lie on the lines $O_{i+1}^{n-1} \cap \ldots \cap O_{i+n-1}^{n-1}$ spanned by the control points $\mathbf{c}_i$ and $\mathbf{c}_{i+1}$ of $\mathbf{s}$. Since the control points $\mathbf{c}_i'$ lie in the hyperplane spanned by $\mathbf{s}'$, we have proved the following theorem.

Theorem 7.5 *If $\mathcal{S}$ contains piecewise constants, then the outer weight points $\mathbf{w}_i$ of a spline $\mathbf{u} \in \mathcal{S}^d$ are the control points of its derivative $\mathbf{u}'$. If $\mathcal{S}'$ also contains piecewise linear splines, then the outer weight points of $\mathbf{u}'$ are the control points of the second derivative $\mathbf{u}''$.*

Remark 7.6 *Let $\mathbf{p}$ be a rational curve with Bézier points $\mathbf{b}_0, \ldots, \mathbf{b}_n$ and (inner) weight points $\mathbf{w}_i = \mathbf{b}_i + \mathbf{b}_{i+1}$. Then, the derivative $\mathbf{p}'$ has the Bézier points $\mathbf{b}_i' := \Delta\mathbf{b}_i := \mathbf{b}_{i+1} - \mathbf{b}_i$ and the inner weight points $\mathbf{w}_i' = \mathbf{b}_i' + \mathbf{b}_{i+1}' = \mathbf{w}_{i+1} - \mathbf{w}_i$. This means that $\mathbf{w}_i'$ forms the intersection of the lines $\mathbf{b}_i\mathbf{b}_{i+1}$ and $\mathbf{w}_i\mathbf{w}_{i+1}$, see Figure 3 for an illustration.*

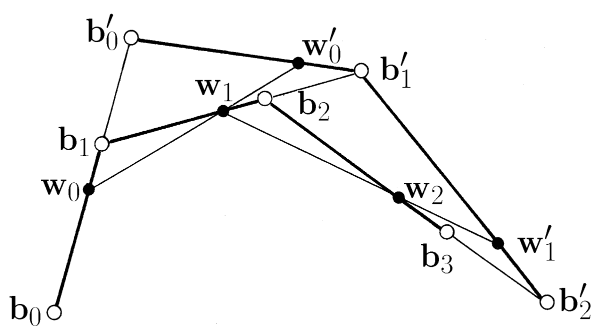

Figure 3: Bézier and weight points of a rational cubic and its derivatives.

Knot insertion means to represent splines of a space $\mathcal{S}$ as elements of a certain larger space $\hat{\mathcal{S}}$, see e.g. [Boe80, Sei92, Maz99]. If we pick a new knot $\hat{a}$ in some knot interval $(\alpha_j, \alpha_{j+1}) = (a_q, a_{q+1})$ and replace in (4.1) the elements of the triples $(\boldsymbol{\alpha}_i, \mu_i, C_i)$ by the element of the triples

$$(\hat{\alpha}_i, \hat{\mu}_i, \hat{C}_i) = \begin{cases} (\boldsymbol{\alpha}_i, \mu_i, C_i) & \text{if } 0 \leq i \leq j \\ (\hat{\boldsymbol{\alpha}}, 1, I) & \text{if } i = j+1 \\ (\boldsymbol{\alpha}_{i-1}, \mu_{i-1}, C_{i-1}) & \text{if } j+2 \leq i \leq k \end{cases}, \tag{7.7}$$

where I is the $n \times n$ identity matrix, and if we replace the ECC-spaces $\mathcal{U}_i$ by the spaces

$$\hat{\mathcal{U}}_i = \begin{cases} \mathcal{U}_i & \text{if } n \leq i \leq j \\ \mathcal{U}_{i-1} & \text{if } j+1 \leq i \leq k+1 \end{cases},$$

then we obtain a spline space $\hat{\mathcal{S}}$ containing $\mathcal{S}$. The new knot $\hat{a}$ could also coincide with an old knot α_j. In this case we require that $\hat{C}_j$ is obtained from C_j by deleting its last row and last column and that the last row of C_j is of the form $[0, \ldots, 0, *]$ to enforce continuity of the osculating flats $O_t^{n-\mu_j-1}$ at $t = \alpha_j$. However, note that a newly inserted connection matrices $\hat{C}_{j+1}$ is the identity matrix see (7.7). Hence, there are no restrictions on the given connection matrices C_i if we insert the same knot $\hat{\alpha} \in (\alpha_j, \alpha_{j+1})$ repeatedly.

Assume that $\mathcal{S}$ and $\hat{\mathcal{S}}$ have bases $\{B_0, \ldots, B_m\}$ and $\{\hat{B}_0, \ldots, \hat{B}_{m+1}\}$ such that supp $B_i = [a_i, a_{i+n+1}]$ and supp $\hat{B}_i = [\hat{a}_i, \hat{a}_{i+n+1}]$. Then, because of Corollary 6.4, any basis function B_i is a linear combination of one or two $\hat{B}_j$, namely

$$B_i = \beta_i \hat{B}_i + \gamma_i \hat{B}_{i+1}, \tag{7.8}$$

where $\gamma_i = 0$ if $i < q - n$, and $\beta_i = 0$ if $i > q - \nu$ with $\nu = \mu_j$ if $\hat{\alpha} = \alpha_j = a_q$ and $\nu = 0$ if $\hat{\alpha} \in (a_q, a_{q+1})$.

Without loss of generality, we assume $\beta_i = 1$ if $\gamma_i = 0$; and $\gamma_i = 1$ if $\beta_i = 0$. Then using (7.8), we obtain for any spline $\mathbf{u} = \sum \mathbf{c}_i B_i$ in $\mathcal{S}^d$ the new representation

$$\mathbf{u} = \sum \hat{\mathbf{c}}_i \hat{B}_i,$$

where

$$\hat{\mathbf{c}}_i = \gamma_{i-1} \mathbf{c}_{i-1} + \beta_i \mathbf{c}_i. \tag{7.9}$$

Geometrically, knot insertion is captured by a projective map $\pi : \mathcal{P}^{m+1} \to \mathcal{P}^m$ that maps a normal curve $\hat{\mathbf{s}}$ of $\hat{\mathcal{S}}$ onto a normal curve $\mathbf{s}$ of $\mathcal{S}$. This interpretation also shows how to compute the weights β_i and γ_i. The new control points are intersections of osculating hyperplanes of $\mathbf{s}$ as in (5.3). In particular, inserting an n-fold knot $\hat{\alpha}$ means that one of the new control

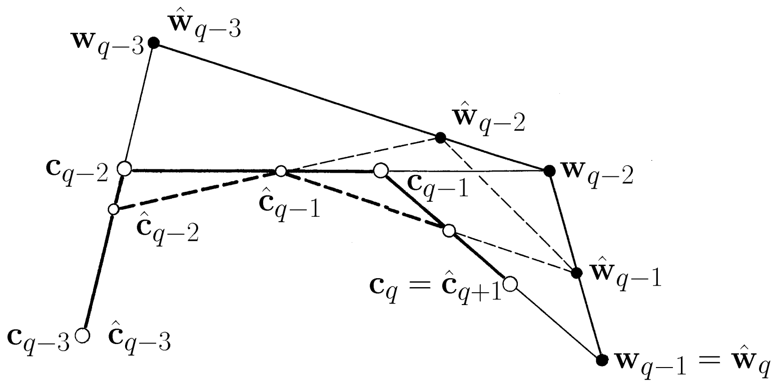

Figure 4: Knot insertion for $n = 3$ and $\nu = 0$.

points equals $\mathbf{s}(\hat{\alpha})$. In other words, $\mathbf{s}$ can be evaluated by repeated knot insertion.

In the sequel, let us assume that $\mathcal{S}$ and thus also $\hat{\mathcal{S}}$ contain piecewise constant functions. So, we can describe the normal curves $\mathbf{s}$ and $\hat{\mathbf{s}}$ by control and weight points, say $\mathbf{c}_i$, $\mathbf{w}_i$ for $\mathbf{s}$ and $\hat{\mathbf{d}}_i$, $\hat{\mathbf{v}}_i$ for $\hat{\mathbf{s}}$. Since $\pi O^{n-1}_{\hat{a}_i}\hat{\mathbf{s}} = O^{n-1}_{\hat{a}_i}\mathbf{s} =: \hat{O}_i$, we get

(7.10)
$$\hat{\mathbf{c}}_i := \pi\hat{\mathbf{d}}_i = \hat{O}_{i+1} \cap \ldots \cap \hat{O}_{i+n} =$$
$$\begin{cases} \mathbf{c}_i & \text{for } i \le q-n \\ (\mathbf{c}_{i-1} \sqcup \mathbf{c}_i) \cap \hat{O}_{q+1} & \text{for } i = q-n+1, \ldots, q-\nu \\ \mathbf{c}_{i-1} & \text{for } i \ge q+1-\nu \end{cases} .$$

Recall that the weight points lie in the hyperplane spanned by the derivative of $\mathbf{s}$ or $\hat{\mathbf{s}}$, respectively. Since $\pi(\text{span } \hat{\mathbf{s}}') = \text{span } \mathbf{s}'$, we get

$$\hat{\mathbf{w}}_i := \pi\hat{\mathbf{v}}_i = \pi\left((\hat{\mathbf{d}}_i \sqcup \hat{\mathbf{d}}_{i+1}) \sqcap \text{span } \hat{\mathbf{s}}'\right) \subseteq (\hat{\mathbf{c}}_i \sqcup \hat{\mathbf{c}}_{i+1}) \sqcap \text{span } \mathbf{s}'.$$

In particular, this means $\hat{\mathbf{w}}_i = \mathbf{w}_i$ for $i \le q-n$ and $\hat{\mathbf{w}}_i = \mathbf{w}_{i-1}$ for $i \ge q-\nu$. The generic construction is illustrated in Figure 4 for $n = 3$.

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