Multivariate Splines with Convex B-Patch Control Nets are Convex

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Abstract: In this paper results from a forthcoming paper are presented concerning the convexity of multivariate spline functions built from B-patches. Conditions are given under which it is possible to define a control net for such spline functions. The control net is understood as a piecewise linear function. If it is convex, then so is the underlying spline.

Keywords: multivariate splines, B-patches, convexity, control nets, Greville-abscissae.

1 Introduction

For the Bézier representation of a bivariate polynomial over some triangle $\triangle$ it is well-known that the convexity of the Bézier net implies the convexity of the polynomial over the triangle $\triangle$. This fact was first proved by Chang and Davis [1984] and later generalized to multivariate polynomials and their Bézier representations over a simplex [DM88, Bes89, Pra95].

Here it is shown that this property is, more generally, even shared by multivariate polynomials and their B-patch representations. Moreover it is also possible to extend the proof to multivariate spline functions and their B-patch control nets.

2 Multivariate B-splines

This paper is based on the B-splines constructed by Dahmen, Micchelli and Seidel [1992] from B-patches. To begin with let us recall the relevant properties and thereby introduce the notation used in this paper:

For any set of **knots** $\mathbf{u}_0, \ldots, \mathbf{u}_k \in \mathbb{R}^s$ or $s \times k{+}1$ matrix $[\mathbf{u}_0 \ldots \mathbf{u}_k]$ the **simplex spline** $M(\mathbf{x}|\mathbf{u}_0 \ldots \mathbf{u}_k)$ is defined as the solution of the functional equation

$$\int_{\mathbb{R}^s} f(\mathbf{x}) M(\mathbf{x}|\mathbf{u}_0 \ldots \mathbf{u}_k) d\mathbf{x} = k! \int_\sigma f([\mathbf{u}_0 \ldots \mathbf{u}_k]\mathbf{t}) d\mathbf{t}$$

for all continuous functions $f(\mathbf{x})$ where

$$\sigma = \{\mathbf{t} \in \mathbb{R}^{k+1} | \mathbf{o} \le \mathbf{t}, |\mathbf{t}| = 1\} \ , \quad |\mathbf{t}| = \text{sum of all coordinates of } \mathbf{t}$$

denotes the **standard** k**-simplex**.

Thus the above normalization implies that

$$\int_{\mathbb{R}^s} M(\mathbf{x}|\mathbf{u}_0 \ldots \mathbf{u}_k) d\mathbf{x} = 1 \ .$$

Now for any $s+1$ knot clusters $\mathbf{u}^\alpha_\beta$, $\alpha = 0,\ldots,s$, $\beta = 0,\ldots,n$, consider the simplices $\sigma_\mathbf{i}$ with vertices $\mathbf{u}^0_{i_0},\ldots,\mathbf{u}^s_{i_s}$ where $\mathbf{i} = (i_0,\ldots,i_s) \in \mathbb{N}_0^{s+1}$ and $|\mathbf{i}| = n$. Then the corresponding splines

$$B_\mathbf{i}(\mathbf{x}) = \frac{\mathrm{vol}_s \sigma_\mathbf{i}}{\binom{n+s}{s}} M(\mathbf{x}|\mathbf{u}^0_0 \ldots \mathbf{u}^0_{i_0} \ldots \mathbf{u}^s_0 \ldots \mathbf{u}^s_{i_s})$$

are the **multivariate B-splines** which were introduced in [DMS92] with the name B-weights.
Throughout the paper we will assume that bold indices $\mathbf{i}, \mathbf{j}, \ldots$ are in $\mathbb{N}_0^{s+1}$ and that the intersection Ω of all simplices $\sigma_\mathbf{i}$, $|\mathbf{i}| \le n$, is non-empty. Then one has the following crucial property:

Theorem 2.1 *Let $p(\mathbf{x})$ be any s-variate polynomial of total degree n and let $p[\mathbf{x}_1 \ldots \mathbf{x}_n]$ be the unique symmetric multiaffine polynomial with the diagonal property $p[\mathbf{x} \ldots \mathbf{x}] = p(\mathbf{x})$. Then for all $\mathbf{x} \in \Omega$ one has*

$$p(\mathbf{x}) = \sum_{|\mathbf{i}|=n} p[\mathbf{u}^0_0 \ldots \mathbf{u}^0_{i_0-1} \ldots \mathbf{u}^s_0 \ldots \mathbf{u}^s_{i_s-1}] B_\mathbf{i}(\mathbf{x}) \ .$$

For the proof one can use the properties of the so-called polar form $p[\mathbf{x}_1 \ldots \mathbf{x}_n]$ and the recurrence relation of simplex splines to evaluate the left and respectively the right hand side of the equation recursively. A comparison then reveals the identity above.
A dimension count futher shows that the $\binom{n+s}{s}$ B-splines $B_\mathbf{i}$ are linearly independent (over Ω).

Remark 2.2 *Theorem 2.1 also shows that for $s = 1$ the $B_i(x)$ are the common univariate B-splines. Further if $\mathbf{u}^\alpha_0 = \cdots = \mathbf{u}^\alpha_n$ for all α, then the $B_\mathbf{i}(\mathbf{x})$ are the truncated Bernstein polynomials over Ω.*

3 Control nets

In order to describe the control net of a polynomial

$$p(\mathbf{x}) = \sum c_\mathbf{i} B_\mathbf{i}(\mathbf{x}) \ , \quad \mathbf{x} \in \Omega \ ,$$

we need the B-spline representation of $\mathbf{x}$. From Theorem 2.1 we obtain

$$\mathbf{x} = \sum \mathbf{x}_\mathbf{i} B_\mathbf{i}(\mathbf{x}) \ , \quad \text{where } \mathbf{x}_\mathbf{i} = \frac{1}{n} \sum_{\alpha=0}^{s} \sum_{\beta=0}^{i_\alpha - 1} \mathbf{u}^\alpha_\beta \ .$$

In particular, if $s = 1$, then the $\mathbf{x}_\mathbf{i}$ are the so-called **Greville abscissa** and if

$$\mathbf{u}^\alpha_\beta = \mathbf{u}^\alpha \quad \text{for all } \alpha \text{ and } \beta$$

then the $\mathbf{x}_\mathbf{i}$ lie on a regular grid, i.e.

$$\mathbf{x}_\mathbf{i} = (i_0 \mathbf{u}^0 + \cdots + i_s \mathbf{u}^s)/n \ .$$

Next we will construct a triangulation whose vertices are the abscissae $\mathbf{x}_\mathbf{i}$ and define the control net of p as the piecewise linear function $c(\mathbf{x})$ which is linear over each simplex of this triangulation and which interpolates the $c_\mathbf{i}$ at the $\mathbf{x}_\mathbf{i}$.

If the $\mathbf{x_i}$ are not too far away from the vertices of a regular grid, then we can obtain a triangulation from a triangulation of the regular grid. Therefore we will first describe a triangulation for the case $\mathbf{u}^\alpha_\beta = \mathbf{u}^\alpha$. Then we change the triangulation by moving the $\mathbf{u}^\alpha_\beta$ independently from each into general positions and present conditions under which the triangulation remains a triangulation with disjoint simplices.
For the construction of a Bézier net Dahmen and Micchelli [1988] used a triangulation due to Allgower and Georg:
Let π be the simplex $\mathbf{u}^0 \dots \mathbf{u}^s$ and ρ the subsimplex whose vertices have the barycentric coordinates

$$\frac{1}{n}\begin{bmatrix} n \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} , \quad \frac{1}{n}\begin{bmatrix} n-1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} , \quad \dots , \frac{1}{n}\begin{bmatrix} n-1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

with respect to π. Let $\mathbf{a}_0, \dots, \mathbf{a}_s$ be these vertices in any arbitrarily fixed order. Counting indices modulo $s+1$ the vertex $\mathbf{a}_0$ and the ordered sequence of vectors

$$\mathbf{v}_0 = \mathbf{a}_1 - \mathbf{a}_0, \dots, \mathbf{v}_s = \mathbf{a}_{s+1} - \mathbf{a}_s$$

describe a simple closed path through all vertices of ρ. Note that $\mathbf{a}_i$ and $\mathbf{v}_i, \dots, \mathbf{v}_{i+s}$ describe the same path. Now if any two successive vectors, say $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ are interchanged, then $\mathbf{a}_i; \mathbf{v}_{i+1}, \dots, \mathbf{v}_{i+s+1}$ describes a path around a simplex ρ' which shares an s–1-dimensional face with ρ. By further transpositions of successive vectors one gets paths around successively adjacent simplices. All the simplices obtained in this way form a triangulation of the entire space $\mathbb{R}^s$. This triangulation is also formed by all hyperplanes spanned by the knot $\mathbf{u}_0$ and any s–1 vectors out of $\{\mathbf{v}_0, \dots, \mathbf{v}_s\}$ and translates of these hyperplanes by integer multiples of the $\mathbf{v}_i$. Thus this triangulation respects the simplex π and can be restricted to π.

Remark 3.1 *If the $\mathbf{a}_i$ denote the vertices of σ in a different order, then the construction above results in a different triangulation.*

4 Conditions on the knot clusters

Assume that all knots in every cluster coincide, i.e. $\mathbf{u}^\alpha_\beta = \mathbf{u}^\alpha$ for all α and β. Then the above triangulation has the following property:

Lemma 4.1 *The union of all simplices with vertex $\mathbf{a}_0$ forms the set of all points*

$$\mathbf{x} = \mathbf{a}_0 + \mu_0\mathbf{v}_0 + \dots + \mu_s\mathbf{v}_s \ , \quad \textit{where } \mu_i \in [0,1] \ .$$

Proof
Let $\mu_0 \geq \dots \geq \mu_s$. Then since $\mathbf{v}_0 + \dots + \mathbf{v}_s = \mathbf{o}$, we can write $\mathbf{x} = \mathbf{a}_0 + \mu_o\mathbf{v}_0 + \dots + \mu_s\mathbf{v}_s$ as

$$\begin{aligned} \mathbf{x} = & (1 - \mu_0 + \mu_s)(\mathbf{a}_0 + \mathbf{v}_0 + \dots + \mathbf{v}_{s-1} + \mathbf{v}_s) \\ & +(\mu_{s-1} - \mu_s)(\mathbf{a}_0 + \mathbf{v}_0 + \dots + \mathbf{v}_{s-1}) \\ & \vdots \\ & +(\mu_0 - \mu_1)(\mathbf{a}_0 + \mathbf{v}_0) \end{aligned}$$

which is a convex combination of the vertices of the simplex given by the loop $\mathbf{a}_0, \mathbf{v}_0 \dots \mathbf{v}_s$. Similarly any ordering of the μ_i corresponds to a loop $\mathbf{a}_0, \mathbf{w}_0 \dots \mathbf{w}_s$ where $(\mathbf{w}_0, \dots, \mathbf{w}_s)$ is a permutation of $(\mathbf{v}_0, \dots, \mathbf{v}_s)$ and vice versa. This completes the proof since all these loops describe all the simplices with vertex $\mathbf{a}_0$. ■

Now we move the $\mathbf{u}_\beta^\alpha$ independently from each other into general positions. This will also change the positions of the $\mathbf{x_i}$ and the shape and positions of the simplices of the triangulation given in Section 3. The new triangulation is still feasible under the following mild restrictions on the knot positions:

Theorem 4.2 *If for all $\alpha = 0, \dots, s$ and $\beta = 0, \dots, n$*

$$\mathbf{u}_\beta^\alpha \in \mathbf{u}^\alpha + [\mathbf{v}_0 \dots \mathbf{v}_s][0, 1/2)^{s+1} ,$$

then any two simplices of the new triangulation have disjoint interiors.

We omit the full proof here and derive only the crucial property on which the proof is based:

$$\begin{aligned}\mathbf{x_i} &= \frac{1}{n}\sum_{\alpha=0}^{s}\sum_{\beta=0}^{i_\alpha-1}\mathbf{u}_\beta^\alpha \\ &\in \frac{1}{n}\sum_{\alpha=0}^{s}\sum_{\beta=0}^{i_\alpha-1}\mathbf{u}^\alpha + [\mathbf{v}_0 \dots \mathbf{v}_s][0,1/2)^{s+1} \\ &= \frac{1}{n}\sum_{\alpha=0}^{s} i_\alpha \mathbf{u}^\alpha + [\mathbf{v}_0 \dots \mathbf{v}_s][0,1/2)^{s+1} .\end{aligned}$$

Thus different $\mathbf{x_i}$ lie in disjoint convex regions.

5 B-patches with convex control nets

Consider the control net of a single B-patch. It is a piecewise linear function defined over some triangulation with the vertices $\mathbf{x_i}$. In general, this triangulation does not form a convex domain for the control net. Therefore we need to explain what is meant by a convex net: First let $\mathbf{q}(\mathbf{x}) = [\mathbf{x} \ q(\mathbf{x})]$ be the graph of a quadratic polynomial $q(\mathbf{x})$ and let $\mathbf{c_i} \in \mathbb{R}^{s+1}$, $|\mathbf{i}| = 2$, be its B-spline control points with respect to the knots $\mathbf{u}_\beta^\alpha$, $\alpha = 0, \dots, s; \beta = 0, 1, 2$, and further let $\mathbf{b_i}$ be the Bézier points of $\mathbf{q}(\mathbf{x})$ over the simplex $\mathbf{u}_0^0 \dots \mathbf{u}_0^s$. Then it follows from Theorem 2.1 that

$$\mathbf{c_i} = \mathbf{b_i} \quad \text{for all } \mathbf{i} \le (1, \dots, 1)$$

and furthermore that the points $\mathbf{b_i}$ and the points $\mathbf{c_i}$, for $\mathbf{i} = \mathbf{e}_i + \mathbf{e}_j$, i fixed, $j = 0, \dots, s$, span the same plane. Thus we have the following property:

Lemma 5.1 *The Bézier and the B-spline control nets of the quadratic polynomial $q(\mathbf{x})$ above are identical over the intersection of their domains.*

Hence we say that the B-spline control net of the quadratic polynomial $p(\mathbf{x})$ is convex if the associated Bézier net of $p(\mathbf{x})$ is convex.

Next consider again a polynomial of degree n

$$p(\mathbf{x}) = \sum_{|\mathbf{i}|=n} c_{\mathbf{i}} B_{\mathbf{i}}(\mathbf{x})$$

given by its B-spline representation over the knot clusters $\mathbf{u}^{\alpha}_{\beta}$, $\alpha = 0, \ldots, s; \beta = 0, \ldots, n$. Let $p[\mathbf{x}_1 \ldots \mathbf{x}_n]$ be the polar form of $p(\mathbf{x})$. Then the quadratic polynomials

$$p_{\mathbf{i}}(\mathbf{x}) = p[\mathbf{x}\ \mathbf{x}\ \mathbf{u}^0_0 \ldots \mathbf{u}^0_{i_0-1} \ldots \mathbf{u}^s_0 \ldots \mathbf{u}^s_{i_s-1}] \ , \quad |\mathbf{i}| = n-2 \ ,$$

have the B-spline representations

$$p_{\mathbf{i}}(\mathbf{x}) = \sum_{|\mathbf{j}|=2} c_{\mathbf{i}+\mathbf{j}} B_{\mathbf{j}}(\mathbf{x})$$

over the knots $\mathbf{u}^{\alpha}_{i_\alpha+\beta}$, $\alpha = 0, \ldots, s; \beta = 0, 1, 2$. Now we can state the main result of this section.

Theorem 5.2 *If the control nets of all quadratic polynomials $p_{\mathbf{i}}(\mathbf{x})$, $|\mathbf{i}| = n-2$, are convex, then $p(\mathbf{x})$ is convex over the intersection Ω of all simplices $\mathbf{u}^0_{i_0} \ldots \mathbf{u}^s_{i_s}$, $|\mathbf{i}| \leq n$.*

Let us sketch the proof: Let $D^2_{\mathbf{v}} f(\mathbf{x})$ be the second derivative of the function f with respect to the direction $\mathbf{v}$. Then one can use, e.g., the multidimensional analog of Proposition 8.2 in [Ram87] to derive

$$D^2_{\mathbf{v}} p(\mathbf{x}) = \frac{n(n-1)}{2} \sum_{|\mathbf{i}|=n-2} (D^2_{\mathbf{v}} p_{\mathbf{i}}) B_{\mathbf{i}}(\mathbf{x}) \ .$$

Since the $p_{\mathbf{i}}$ have a convex Bézier net, they are convex functions, see e.g. [DM88]. Hence the second directional derivatives $D^2_{\mathbf{v}} p_{\mathbf{i}}$ are non-negative which implies that $D^2_{\mathbf{v}} p(\mathbf{x})$ is non-negative and thus the convexity of $p(\mathbf{x})$ over Ω.

6 Splines with convex control nets

The results above for a single B-patch can be extended to splines:
Let $\mathbf{u}^{\alpha}$, $\alpha \in \mathbb{Z}$, be the vertices of some triangulation $\mathcal{T}$ covering the entire space $\mathbb{R}^s$. Here we think of $\mathcal{T}$ as a subset of $\mathbb{Z}^{s+1}$ such that the simplices $\mathbf{u}^{a_0} \ldots \mathbf{u}^{a_s}$, $\mathbf{a} = (a_0, \ldots, a_s) \in \mathcal{T}$ form the triangulation. In the following we will always assume that $\mathcal{T}$ contains each simplex only once, i.e. for any $\mathbf{a} \in \mathcal{T}$ there is no other permutation of $\mathbf{a}$ in $\mathcal{T}$. Further let $\mathbf{u}^{\alpha}_{\beta}$, $\beta = 0, \ldots, n$ be associated knot clusters and assume that the intersections $\Omega_{\mathbf{a}}$ of all simplices $\mathbf{u}^{a_0}_{i_0} \ldots \mathbf{u}^{a_s}_{i_s}$, $|\mathbf{i}| \leq n$, are non-empty for all $\mathbf{a} \in \mathcal{T}$. Then consider the spline

$$s(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{T}} \sum_{|\mathbf{i}|=n} c^{\mathbf{a}}_{\mathbf{i}} B^{\mathbf{a}}_{\mathbf{i}}(\mathbf{x})$$

where $B^{\mathbf{a}}_{\mathbf{i}}$ is the B-spline over the knots $\mathbf{u}^{\alpha}_{\beta}$, $\alpha = a_0, \ldots, a_s; \beta = 0, \ldots, i_\alpha$. In order to define the control net of $s(\mathbf{x})$ as a piecewise linear function we need the abscissae

$$\mathbf{x}^{\mathbf{a}}_{\mathbf{i}} = \frac{1}{n} \sum_{\alpha=a_0,\ldots,a_s} \sum_{\beta=0}^{i_\alpha} \mathbf{u}^{\alpha}_{\beta} \ .$$

Then for each $\mathbf{a} \in \mathcal{T}$ we construct a triangulation having the abscissae $\mathbf{x}_{\mathbf{i}}^{\mathbf{a}}$ as vertices as described in Section 4 using the loops

$$\mathbf{v}_0^{\mathbf{a}} = \mathbf{u}^{a_1} - \mathbf{u}^{a_0} \quad , \quad \ldots \quad , \quad \mathbf{v}_s^{\mathbf{a}} = \mathbf{u}^{a_0} - \mathbf{u}^{a_s} \quad .$$

In order to obtain a correct triangulation of all $\mathbf{x}_{\mathbf{i}}^{\mathbf{a}}$, $\mathbf{a} \in \mathcal{T}$, we need to restrict the positions of the knots. Such a condition is given by the following extension of Theorem 4.2:

Theorem 6.1 *Let Ω_α be the intersections*

$$\Omega_\alpha = \cap\{[\mathbf{v}_0^{\mathbf{a}} \ldots \mathbf{v}_s^{\mathbf{a}}][0,1/2)^{s+1} \mid \mathbf{a} \in \mathcal{T}, \ \alpha \textit{ is a coordinate of } \mathbf{a}\}$$

and for all $\alpha \in \mathbb{Z}$ and $\beta = 0, \ldots, n$ let $\mathbf{u}_\beta^\alpha \in \mathbf{u}^\alpha + \Omega_\alpha$. Then any two simplices of the triangulation of the $\mathbf{x}_{\mathbf{i}}^{\mathbf{a}}$, $\mathbf{a} \in \mathcal{T}$, $|\mathbf{i}| = n$, have disjoint interiors.

Theorem 6.1 enables us to define the **control net** of $s(\mathbf{x})$ as the piecewise linear function which is composed of the control nets of the patches

$$s_{\mathbf{a}}(\mathbf{x}) = \sum_{|\mathbf{i}|=n} c_{\mathbf{i}}^{\mathbf{a}} B_{\mathbf{i}}^{\mathbf{a}}(\mathbf{x}) \quad , \quad \mathbf{x} \in \Omega_{\mathbf{a}} \quad .$$

Note that the control nets of the patches over the sets $\Omega_{\mathbf{a}}$ are always continuous, but the entire control net of $s(\mathbf{x})$ is continuous only if $c_{\mathbf{i}}^{\mathbf{a}} = c_{\mathbf{j}}^{\mathbf{b}}$ whenever $\mathbf{x}_{\mathbf{i}}^{\mathbf{a}} = \mathbf{x}_{\mathbf{j}}^{\mathbf{b}}$. Now, for this control net of $s(\mathbf{x})$ we can state the main result presented in this paper:

Theorem 6.2 *Let the control net of $s(\mathbf{x})$ be continuous and such that the subnets for all patches $s_{\mathbf{a}}(\mathbf{x})$, $\mathbf{x} \in \Omega_{\mathbf{a}}$, satisfy the conditions of Theorem 5.2. Then the spline function $s(\mathbf{x})$ is convex for all $\mathbf{x} \in \mathbb{R}$.*

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